International Journal of Systems Science, 2002, volume 33, number 12, pages 995–1012

Globally stable model reference adaptive control based on fuzzy description of the plant

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A novel fuzzy adaptive control algorithm is presented that belongs to direct model reference adaptive techniques based on a fuzzy (Takagi–Sugeno) model of the plant. The global stability of the overall system is proven, namely all the signals in the system remain bounded while the tracking error and estimated parameters converge to some residual set that depends on the size of disturbance and high-order parasitic dynamics. The hallmarks of the approach are its simplicity and transparency. The proposed algorithm is a straightforward extension of classical model reference adaptive control (MRAC) with a robust adaptive law to nonlinear systems described by fuzzy models. The performance of the approach was tested on a simulated plant and compared with the performance of a PI controller and a classical MRAC.

1. Introduction

Model reference adaptive techniques became very popular in the 1960s and 1970s due to global stability that was guaranteed by the Lyapunov redesign (e.g. Monopoli 1974, Narendra *et al.* 1980) of the earlier gradient adaptation methods. However, those adaptive methods never went beyond the theory of the academic literature. As recognized in the 1980s, their main drawback was sensitivity to unmodelled dynamics and disturbances (Rohrs *et al.* 1985). Robust adaptive controllers (e.g. Ioannou and Datta 1991, Ioannou and Sun 1996) overcome this drawback, but practical applications still seem to be missing.

In our opinion, the main reason for the lack of practical applications is the nonlinearity of real plants. Classical adaptive systems (in this paper, adaptive systems for LTI plants that were developed by the end of the 1970s, e.g. Narendra *et al.* 1980, are referred to as classical) can adjust their parameters to different operating points but this procedure takes some time, and if the operating conditions change frequently, the adaptive system is unable to track them. Besides, from a theoretical viewpoint, the unknown plant parameters are assumed to be constant, so the application of such adaptive systems to plants with time-varying parameters is questionable. On the other hand, if the nonlinearity of the plant is included into the unmodelled part of the linear plant, catered for by the robustness properties of the controller, the performance of the closed-loop system becomes poor and all advantages of the adaptive controllers over classical fixed-gain robust controllers are lost.

Classical adaptive control was extended in the 1980s and the 1990s to the time-varying (Tsakalis and Ioannou 1987) and nonlinear plants (Krstić *et al.* 1995). Since we restricted our attention mainly to nonlinear plants that were more or less time-invariant, the former approaches were not so relevant even though they produced better results than classical adaptive control. The main drawback of adaptive control algorithms for nonlinear plants is that they demand fairly good knowledge of mathematics and are thus avoided by practising engineers.

In the last decade fuzzy controllers have proven themselves capable of coping with plant nonlinearities, not only theoretically, but also by practical applications (Škrjanc and Matko 2000). Much effort has been put to fuzzy gain-scheduling (Tzafestas *et al.* 2001). Many different approaches to neuro-fuzzy adaptive control have also been reported. Jagannathan *et al.* (1994) studied the tracking performance of model reference adaptive control (MRAC) using multilayer neural networks based on a Lyapunov stability approach. Škrjanc *et al.* (1997) and Škrjanc and Matko (1997) proposed an

Received 28 June 2001. Revised 5 August 2002. Accepted 20 September 2002.

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International Journal of Systems Science ISSN 0020–7721 print/ISSN 1464–5319 online © 2002 Taylor & Francis Ltd
http://www.tandf.co.uk/journals
DOI: 10.1080/0020772021000046252

indirect fuzzy adaptive control algorithm. Wang (1993) and Spooner and Passino (1996) presented stable adaptive fuzzy control for nonlinear plants. Hu and Lu (1998) proposed the adaptive observer and the nonlinear controller which are based on neural networks. The fuzzy learning approach is quite common in mobile robot motion control where Rigatos *et al.* (2000) proposed the approach with fuzzy membership functions being adapted. The sliding-mode fuzzy-logic controller was realized as a fuzzy learning automaton by Rigatos *et al.* (2001).

The main drawback of fuzzy (adaptive) controllers is the lack of theoretical background about the closed-loop stability. An attempt to join the stability issues of model reference adaptive systems with the capability of fuzzy systems to cope with nonlinear plants is given in the present paper. The direct fuzzy model reference adaptive control (DFMRAC) algorithm is introduced. The model of the plant is given in the simple fuzzy form (Takagi and Sugeno 1985). The main idea of the approach is fuzzification of estimated parameters (control gains) resulting in the control and the adaptive laws that very much resemble the classical MRAC. The resulting equations are very simple since the plant is assumed to be predominantly of the first order (but nonlinear), while the parasitic high-order dynamics are included in the non-modelled part and do not cause instability due to the robust adaptive law. In our opinion such restriction is not too stringent since plants that belong to this class occur quite often in process industries. Note that good results are still obtained even in the case where parasitic dynamics are not neglectable.

The stability of the DFMRAC is examined thoroughly in the framework proposed by Ioannou and Sun (1996). The boundedness of estimated parameters, the tracking error and all the signals in the system are proven as well as the convergence of the tracking error and estimated parameters to some residual set that depends on the size of the disturbance and the parasitic dynamics.

The paper is organized as follows. In Section 2, the class of plants that will be discussed is presented. In Section 3, the description of the proposed algorithm is given. The performance of the algorithm is tested on a simulated plant in Section 4. The conclusions are presented in Section 5. In the appendices the proofs of the important theorems are given together with the necessary background.

2. Class of plants taken into account

In the literature there are many approaches to nonlinear system identification. Among them identification by the use of fuzzy models is quite common. Since our aim was to use simple algorithms, the Takagi–Sugeno model was chosen to describe the plant behaviour (Takagi and Sugeno 1985). If the first-order plant is assumed and the nonlinearity of the plant depends on two measurable quantities, z_1 and z_2, the model is described by k if-then rules of the following form:

$$\begin{aligned}&\textbf{if } z_1 \textbf{ is } \mathbf{A}_{i_a} \textbf{ and } z_2 \textbf{ is } \mathbf{B}_{i_b} \textbf{ then } \dot{y}_p = -a_i y_p + b_i u\\ &i_a = 1,\dots,n_a;\ i_b = 1,\dots,n_b;\ i = 1,\dots,k,\end{aligned} \tag{1}$$

where u and y_p are the input and the output of the plant, respectively, $\mathbf{A}_{i_a}$ and $\mathbf{B}_{i_b}$ are fuzzy membership functions, and a_i and b_i are the plant parameters in the i-th fuzzy domain. The antecedent variables that define in which fuzzy domain the system is currently situated are denoted by z_1 and z_2 (actually, there can be only one such variable and there can also be more, but this does not affect the approach described here). There are n_a and n_b membership functions for the first and the second antecedent variable, respectively. The product $k = n_a \times n_b$ defines the number of fuzzy rules. The membership functions have to cover the whole operating area of the system. The output of the Takagi–Sugeno model is then given by the following equation:

$$\dot{y}_p = \frac{\sum_{i=1}^{k}(\beta_i^0(\varphi)(-a_i y_p + b_i u))}{\sum_{i=1}^{k}\beta_i^0(\varphi)}, \tag{2}$$

where φ is the vector of antecedent variables z_i. The degree of fulfilment $\beta_i^0(\varphi)$ is obtained using T-norm, which in this case is a simple algebraic product of membership functions:

$$\beta_i^0(\varphi) = T(\mu_{A_{i_a}}(z_1), \mu_{B_{i_b}}(z_2)) = \mu_{A_{i_a}}(z_1)\cdot\mu_{B_{i_b}}(z_2), \tag{3}$$

where $\mu_{A_{i_a}}(z_1)$ and $\mu_{B_{i_b}}(z_2)$ are degrees of fulfilment of the corresponding membership functions. The degrees of fulfilment for the whole set of rules can be written in the compact form

$$\boldsymbol{\beta}^0 = [\beta_1^0 \quad \beta_2^0 \quad \dots \quad \beta_k^0]^{\mathrm{T}} \tag{4}$$

and given in normalized form as

$$\boldsymbol{\beta} = \frac{\boldsymbol{\beta}^0}{\sum_{i=1}^{k}\beta_i^0}. \tag{5}$$

Owing to (2) and (5), the first-order plant can be modelled in fuzzy form as

$$\dot{y}_p = -(\boldsymbol{\beta}^{\mathrm{T}}\mathbf{a})y_p + (\boldsymbol{\beta}^{\mathrm{T}}\mathbf{b})u, \tag{6}$$

where

$$\mathbf{a} = [a_1 \quad a_2 \quad \cdots \quad a_k]^{\mathrm{T}} \quad \text{and} \quad \mathbf{b} = [b_1 \quad b_2 \quad \cdots \quad b_k]^{\mathrm{T}}$$

are vectors of unknown plant parameters in respective fuzzy domains.

To assume that the controlled system is of the first order is quite a huge idealization; therefore, parasitic dynamics are included in the model of the plant. A linear time-invariant system of the first order with stable factor plant perturbations is described by the following equation:

$$y_p(s) = \frac{\dfrac{b}{s+c} + \Delta_1(s)}{\dfrac{s+a}{s+c} + \Delta_2(s)}\, u(s), \tag{7}$$

where $\dfrac{b}{s+a}$ is the transfer function of the nominal system, c is a positive constant, $\Delta_1(s)$ and $\Delta_2(s)$ are stable transfer functions (Vidyasagar 1993), and $u(s)$ and $y_p(s)$ are the Laplace transforms of the plant's input and output, respectively. By multiplying numerator and denominator of (7) by $(s+c)$ the following is obtained:

$$y_p(s) = \frac{b + \Delta'_u(s)}{s + a + \Delta'_y(s)}\, u(s), \tag{8}$$

where the definition of $\Delta'_u(s)$ and $\Delta'_y(s)$ follows directly. Since a and b in (8) are not known, they can be found such that $\Delta'_u(s)$ and $\Delta'_y(s)$ are definitely strictly proper transfer functions (if they are only biproper, a solution with different a, b, $\Delta'_u(s)$ and $\Delta'_y(s)$ can always be found such that $\Delta'_u(s)$ and $\Delta'_y(s)$ are strictly proper and (8) still holds). The equation (8) can be rewritten as

$$sy_p = -ay_p + bu - \Delta'_y(s)y_p + \Delta'_u(s)u. \tag{9}$$

By taking into account the fuzzy model of the plant (6), the first two terms in (9) that apply to linear systems are replaced and the plant model becomes:

$$\dot{y}_p = -(\boldsymbol{\beta}^{\mathrm{T}}\mathbf{a})y_p + (\boldsymbol{\beta}^{\mathrm{T}}\mathbf{b})u - \Delta'_y(p)y_p + \Delta'_u(p)u, \tag{10}$$

where p is a differential operator $\mathrm{d}/\mathrm{d}t$, while $\Delta'_y(p)$ and $\Delta'_u(p)$ are linear operators in the time domain that are equivalent to transfer functions $\Delta'_y(s)$ and $\Delta'_u(s)$. It is assumed that the plant is also disturbed by an external disturbance and the final model of the plant used in this paper is obtained by adding the disturbance d' to (10):

$$\dot{y}_p = -(\boldsymbol{\beta}^{\mathrm{T}}\mathbf{a})y_p + (\boldsymbol{\beta}^{\mathrm{T}}\mathbf{b})u - \Delta'_y(p)y_p + \Delta'_u(p)u + d'. \tag{11}$$

Assumptions on the plant model (11):

A1. Absolute values of the elements of vector **b** are bounded from below and from above: $b_{\min} < |b_i| < b_{\max}$, $i = 1, 2, \ldots, k$ and $b_{\min}$ and $b_{\max}$ are some positive constants.

A2. Absolute values of the elements of vector **a** are bounded from above: $|a_i| < a_{\max}$, $i = 1, 2, \ldots, k$ and $a_{\max}$ is a positive constant.

A3. Signs of the elements in vector **b** are the same.

If some b_i approached 0, the system would become almost uncontrollable in that operating point. We know that uncontrollability is not easily circumvented in any type of control, especially not in adaptive control; therefore the first part of **A1** ('bounded from below' part) has to hold. The same goes for the consequences of violation of the assumption **A3**, namely some operating points (characterized by $\boldsymbol{\beta}$) would exist where the gain of the linearized plant $\boldsymbol{\beta}^{\mathrm{T}}\mathbf{b}$ was 0 if the elements in **b** were not of the same sign. Also, the gain of the linearized plant would be positive in some operating points and negative in others. The control of such a plant would always be a problem and our attention is not directed to plants of a kind. Although fuzzy models can be regarded as universal approximators, only arbitrary small modelling errors are attainable in general. That is why too large elements of **a** or **b** would cause large modelling errors (the second part of **A1**—'bounded from above' part—and **A2** have to hold).

It is worth mentioning that only dominant plant dynamics are assumed nonlinear while parasitic dynamics are linear. This is not a too unrealistic assumption since only the upper bound on the certain norms of the unmodelled dynamics are used in the theorem given later. If the nonlinearity of the unmodelled dynamics is not too obvious, the proposed plant model is sufficient and it can be used in quite a broad spectre of real plants, especially in process industries where first-order nonlinear systems are quite common.

The prerequisite for using model (11) is that we know what system variables the nonlinearity depends upon, i.e. what signals (z_1 and z_2 in this section) influence the calculation of $\boldsymbol{\beta}$. The choice of these, so called fuzzification or antecedent variables, depends on the plant behaviour and is a similar problem to that of structural identification (Takagi and Sugeno 1985) in the case of Takagi–Sugeno model. In Takagi and Sugeno (1985) it was proposed that these variables were the system input and output. Since the realization of the control is not possible if $\boldsymbol{\beta}$ depends on u, $\boldsymbol{\beta}$ has to be calculated by the use of y_p and/or some other signal(s) that might be correlated with the change of the system dynamics. Since the choice of fuzzification variables does not influence the form of the model (11) and the algorithm proposed below, it will not be addressed here.

3. Proposed direct fuzzy model reference adaptive control algorithm

The model of the plant was described above. The first two terms on the right-hand side of (11) will serve as a model for control design while the other terms will be catered for by the robustness properties of the adaptive and control laws since they are unknown in advance. Note that **a** and **b** are also unknown. To overcome this difficulty adaptive control will be used.

The question still remains whether to use direct or indirect adaptive scheme. Both approaches have advantages and disadvantages that are well known and documented for adaptive control of LTI plants (e.g. Ioannou and Sun 1996). Since it is our belief that it is much harder to prove the global stability in the latter case, direct adaptive control was used in our approach, i.e. control parameters were estimated directly by using measurable signals. The task of this section is to find the control and adaptive laws that suit the design objective.

It was mentioned that the proposed approach to fuzzy adaptive control resembles very much the classical MRAC. Since our attention is focused on plants that are dominantly of the first order, MRAC of the first-order LTI plant will be recalled first. Later on, the control algorithm will be extended to nonlinear plants of the first order with high-order parasitics.

3.1. *MRAC of LTI plants*

Let us briefly recall the classical approach to MRAC of the first-order linear time invariant system. The approach described below is based on Lyapunov theory and can be found in most textbooks on adaptive control (e.g. Åstöm and Wittenmark 1995).

The LTI plant of the first order can be described by the differential equation

$$\dot{y}_p = -a y_p + b u, \tag{12}$$

where u and y_p are the input and output of the plant, respectively, while a and b are unknown constants. By choosing reference model

$$\dot{y}_m = -a_m y_m + b_m w \tag{13}$$

a control law

$$u = f w - q y_p \tag{14}$$

follows to achieve the design objective where w is the reference signal. The classical solution to find the correct values for control parameters f and q is to estimate them by the following adaptive law:

$$\begin{aligned} \dot{f} &= -\gamma_f \operatorname{sgn}(b) e w \\ \dot{q} &= \gamma_q \operatorname{sgn}(b) e y_p, \end{aligned} \tag{15}$$

where e is the tracking error, defined as the difference between y_p and y_m, while γ_f and γ_q are arbitrary positive constants, usually referred to as adaptive gains.

As shown by Rohrs *et al.* (1985), the above approach is not robust with respect to high-order unmodelled dynamics and disturbances, therefore the adaptive law or the control law or external excitation has to be changed to achieve the robustness. As will be shown below, our approach was to use the modified adaptive law.

3.2. *DFMRAC for the class of nonlinear plants*

The reason for presenting MRAC for the first-order linear plant above is that the proposed DFMRAC algorithm is a straightforward extension of the former. The latter assumes the fuzzification of the forward gain f and the feedback gain q. The fuzzified gains are described by means of fuzzy numbers $\mathbf{f}$ and $\mathbf{q}$

$$\begin{aligned} \mathbf{f}^{\mathrm{T}} &= [\, f_1 \quad f_2 \quad \cdots \quad f_k \,] \\ \mathbf{q}^{\mathrm{T}} &= [\, q_1 \quad q_2 \quad \cdots \quad q_k \,], \end{aligned} \tag{16}$$

where k is the number of fuzzy rules as mentioned above. The reference model is the same as in (13)

$$\dot{y}_m = -a_m y_m + b_m w. \tag{17}$$

The control law is obtained by slightly extending (14), namely scalar control gains are substituted by the vector ones:

$$u = (\boldsymbol{\beta}^{\mathrm{T}} \mathbf{f}) w - (\boldsymbol{\beta}^{\mathrm{T}} \mathbf{q}) y_p. \tag{18}$$

The tracking error is the same as before

$$e = y_p - y_m. \tag{19}$$

3.2.1. *Adaptive law*

The most important part of the algorithm is the adaptive law that can be put down in the scalar form

$$\begin{aligned} \dot{f}_i &= -\gamma_{fi} b_{\mathrm{sign}} \varepsilon w \beta_i - \gamma_{fi} |\varepsilon m| \nu_0 f_i \beta_i \quad i = 1, 2, \ldots, k \\ \dot{q}_i &= \gamma_{qi} b_{\mathrm{sign}} \varepsilon y_p \beta_i - \gamma_{qi} |\varepsilon m| \nu_0 q_i \beta_i \quad i = 1, 2, \ldots, k, \end{aligned} \tag{20}$$

or in equivalent vector form which is more suitable for analysis due to its compactness

$$\begin{aligned} \dot{\mathbf{f}} &= -\boldsymbol{\Gamma}_f b_{\mathrm{sign}} \varepsilon w \boldsymbol{\beta} - \boldsymbol{\Gamma}_f |\varepsilon m| \nu_0 \mathbf{F} \boldsymbol{\beta} \\ \dot{\mathbf{q}} &= \boldsymbol{\Gamma}_q b_{\mathrm{sign}} \varepsilon y_p \boldsymbol{\beta} - \boldsymbol{\Gamma}_q |\varepsilon m| \nu_0 \mathbf{Q} \boldsymbol{\beta}, \end{aligned} \tag{21}$$

where γ_{fi} and γ_{qi} are positive scalar adaptive gains, ε is the error that will be defined below, m is a variable for normalization to be defined, ν_0 is a design parameter that determines the influence of the 'leakage' (Ioannou and Sun 1996), $\mathbf{F} = \mathrm{diag}(\mathbf{f})$, $\mathbf{Q} = \mathrm{diag}(\mathbf{q})$, and $\boldsymbol{\Gamma}_f$ and $\boldsymbol{\Gamma}_q$ are diagonal matrices of the corresponding adaptive gains γ_{fi} and γ_{qi}, respectively. If the sign of the elements in vector $\mathbf{b}$ in (11) is negative, b_{sign} is -1, otherwise it is $+1$. By introducing $\boldsymbol{\theta}^{\mathrm{T}} \triangleq [\mathbf{f}^{\mathrm{T}} \quad \mathbf{q}^{\mathrm{T}}]$ and $\boldsymbol{\omega}^{\mathrm{T}} \triangleq [\boldsymbol{\beta}^{\mathrm{T}} w \quad -\boldsymbol{\beta}^{\mathrm{T}} y_p]$, (21) can be made even more compact

$$\dot{\boldsymbol{\theta}} = -\boldsymbol{\Gamma} b_{\mathrm{sign}} \varepsilon \boldsymbol{\omega} - \boldsymbol{\Gamma} |\varepsilon m| \nu_0 \boldsymbol{\theta}_d \boldsymbol{\beta}, \tag{22}$$

where $\boldsymbol{\Gamma}$ is the diagonal matrix of scalar adaptive gains and $\boldsymbol{\theta}_d^{\mathrm{T}} = [\mathbf{F}^{\mathrm{T}} \quad \mathbf{Q}^{\mathrm{T}}]$.

There are some remarks concerning the adaptive law (22) that have to be mentioned. The first term on the right-hand side of (22) is equivalent to the adaptive law (15). The second term introduces leakage, more specifically so-called e_1-modification (Narendra and Annaswamy 1987). Note that instead of the product $\boldsymbol{\theta}_d\boldsymbol{\beta}$, only $\boldsymbol{\theta}$ is used in Narendra and Annaswamy, where the situation was simpler since the plant was LTI. The difference is seen more clearly from (20). When the system leaves a certain operating region (fuzzy domain), the corresponding membership function β_i becomes 0. If β_i was not included in the second term on the right-hand side of (20), the system would gradually forget estimated parameter values f_i and q_i. When the system returned to the operating region, it would use the wrong parameter estimates. By including β_i in the second term on the right-hand side of (20) the adaptation of the respective parameter freezes until β_i is non-zero. This difference makes the analysis of the properties of adaptive law a little different than the one performed by Ioannou and Sun (1996). On the other hand, the classical demand on the excitation of the external signal that prevents parameter drift is relaxed a little since some parameters are frozen at each instant and only those that correspond to the current fuzzy domains are potential candidates for the undesired adaptation (parameter drift).

Note that the adaptation is not governed by the tracking error e in (22). Instead, signal ε is used which is defined as

$$\varepsilon = e - G_m(p)(\varepsilon n_s^2), \tag{23}$$

where $n_s^2 = m^2 - 1$ and $G_m(p)$ is the reference model operator in the time domain.

Theorem 1: *Adaptive law described by* (20) (*or equivalently* 21 *or* 22) *guarantees boundedness of the estimated parameter vectors* $\mathbf{f}$ *and* $\mathbf{q}$ *provided that m is designed such that*

$$\frac{w}{m}, \frac{y_p}{m} \in \mathcal{L}_\infty. \tag{24}$$

Proof: Lyapunov-like function is chosen

$$V_{fi} = \frac{1}{2\gamma_{fi}} f_i^2. \tag{25}$$

Its derivative is:

$$\begin{aligned} \dot{V}_{fi} &= \frac{1}{\gamma_{fi}} f_i \dot{f}_i = -b_{\text{sign}} \varepsilon w f_i \beta_i - |\varepsilon m| \nu_0 f_i^2 \beta_i \\ &= -|\varepsilon m| \nu_0 \beta_i \left(f_i^2 + b_{\text{sign}} \operatorname{sgn}(\varepsilon m) \frac{w}{m} \frac{1}{\nu_0} f_i \right). \end{aligned} \tag{26}$$

The derivative of the Lyapunov-like function (25) is non-positive if

$$|f_i| > \left| b_{\text{sign}} \operatorname{sgn}(\varepsilon m) \frac{w}{m} \frac{1}{\nu_0} \right| = \frac{1}{\nu_0} \left| \frac{w}{m} \right|. \tag{27}$$

Since $w/m \in \mathcal{L}_\infty$, $|f_i|$ is also bounded from above (it decreases until it reaches $1/\nu_0 |w(t)/m(t)|$). In a similar manner it can be shown that q_i is bounded if $y_p/m \in \mathcal{L}_\infty$. Since the design of m is at the discretion of the designer, it can be concluded that estimated parameters are bounded, i.e. $\mathbf{f}$, $\mathbf{q} \in \mathcal{L}_\infty$. □

3.2.2. *Error model*

By subtracting (17) from (11), the following equation is obtained:

$$\begin{aligned} \dot{e} = -a_m e &+ [(\boldsymbol{\beta}^{\mathrm{T}}\mathbf{b})(\boldsymbol{\beta}^{\mathrm{T}}\mathbf{f}) - b_m]w - [(\boldsymbol{\beta}^{\mathrm{T}}\mathbf{b})(\boldsymbol{\beta}^{\mathrm{T}}\mathbf{q}) \\ &+ (\boldsymbol{\beta}^{\mathrm{T}}\mathbf{a}) - a_m]y_p + \Delta_u'(p)u - \Delta_y'(p)y_p + d'. \end{aligned} \tag{28}$$

It is impossible to find such $\mathbf{f}$ and $\mathbf{q}$ that would make the expressions in brackets equal to zero for a general case. This means that the perfect tracking of the reference model is not possible by any choice of the control vectors even in the case when no parasitic dynamics or disturbances are present. A decision has to be made about what values for the elements of the vectors $\mathbf{f}$ and $\mathbf{q}$ are the desired ones. Those elements will be denoted by f_i^* and q_i^*. They will be obtained by making the expressions in brackets in (28) equal to zero:

$$\begin{aligned} (\boldsymbol{\beta}^{\mathrm{T}}\mathbf{b})(\boldsymbol{\beta}^{\mathrm{T}}\mathbf{f}) - b_m &= 0 \\ (\boldsymbol{\beta}^{\mathrm{T}}\mathbf{b})(\boldsymbol{\beta}^{\mathrm{T}}\mathbf{q}) + (\boldsymbol{\beta}^{\mathrm{T}}\mathbf{a}) - a_m &= 0. \end{aligned} \tag{29}$$

As established before, a general solution for $\mathbf{f}$ and $\mathbf{q}$ in (29) does not exist. A particular solution will be found for the cases where only one fuzzy domain is activated. This is done for all k fuzzy domains to obtain all f_i^*'s and q_i^*'s. Mathematically, this is done by setting

$$\boldsymbol{\beta} = [0 \quad \cdots \quad 0 \quad 1 \quad 0 \quad \cdots \quad 0]^{\mathrm{T}}$$

in (29), i.e. by choosing i-th element of the vector $\boldsymbol{\beta}$ equal to 1 while others are equal to 0:

$$\begin{aligned} b_i f_i^* - b_m &= 0 \quad i = 1, 2, \ldots, k, \\ b_i q_i^* + a_i - a_m &= 0 \quad i = 1, 2, \ldots, k. \end{aligned} \tag{30}$$

This actually means that the desired control parameters are the same as they would be if obtained in each fuzzy domain separately. This also leads to the prefect tracking if the plant is currently in only one fuzzy domain (local linear model of that domain applies) and there is no parasitic dynamics or disturbances. If some of the above conditions are violated, some terms on the right-hand side of (28) are non-zero. It will be shown that these terms do not affect the stability of the system.

Desired control parameters

$$\begin{aligned} \mathbf{f}^{*\mathrm{T}} &= [f_1^* \quad f_2^* \quad \cdots \quad f_k^*] \\ \mathbf{q}^{*\mathrm{T}} &= [q_1^* \quad q_2^* \quad \cdots \quad q_k^*] \end{aligned} \tag{31}$$

are bounded due to (30) and the assumptions **A1** and **A2**. The parameter errors are defined as:

$$\tilde{\mathbf{f}} = \mathbf{f} - \mathbf{f}^* \qquad \tilde{\mathbf{q}} = \mathbf{q} - \mathbf{q}^*. \tag{32}$$

Our wish is to change the expressions in the brackets of (28) with new ones

$$(\boldsymbol{\beta}^{\mathrm{T}}\mathbf{b})(\boldsymbol{\beta}^{\mathrm{T}}\mathbf{f}) - b_m = b\tilde{\mathbf{f}}^{\mathrm{T}}\boldsymbol{\beta} + b_m\frac{\eta_w}{w}$$
$$(\boldsymbol{\beta}^{\mathrm{T}}\mathbf{b})(\boldsymbol{\beta}^{\mathrm{T}}\mathbf{q}) + (\boldsymbol{\beta}^{\mathrm{T}}\mathbf{a}) - a_m = b\tilde{\mathbf{q}}^{\mathrm{T}}\boldsymbol{\beta} + b_m\frac{\eta_y}{y_p}, \tag{33}$$

where

$$b = \inf_{\boldsymbol{\beta}} \boldsymbol{\beta}^{\mathrm{T}}\mathbf{b} = \min_i b_i. \tag{34}$$

By using (32) the first equation in (33) yields:

$$b\tilde{\mathbf{f}}^{\mathrm{T}}\boldsymbol{\beta} + b_m\frac{\eta_w}{w} = \boldsymbol{\beta}^{\mathrm{T}}\mathbf{b}\mathbf{f}^{\mathrm{T}}\boldsymbol{\beta} - b_m = \boldsymbol{\beta}^{\mathrm{T}}\mathbf{b}\mathbf{f}^{*\mathrm{T}}\boldsymbol{\beta} + \boldsymbol{\beta}^{\mathrm{T}}\mathbf{b}\tilde{\mathbf{f}}^{\mathrm{T}}\boldsymbol{\beta} - b_m. \tag{35}$$

Define matrix $\mathbf{B}$:

$$\mathbf{B} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \end{bmatrix} [b_1^{-1} \quad b_2^{-1} \quad \cdots \quad b_k^{-1}] = \begin{bmatrix} 1 & \frac{b_1}{b_2} & \cdots & \frac{b_1}{b_k} \\ \frac{b_2}{b_1} & 1 & \cdots & \frac{b_2}{b_k} \\ \vdots & \vdots & \ddots & \\ \frac{b_k}{b_1} & \frac{b_k}{b_2} & & 1 \end{bmatrix}. \tag{36}$$

Using $[1 \quad 1 \quad \cdots \quad 1]\boldsymbol{\beta} = 1$ (see equation 5) and (36), equation (35) yields:

$$b\tilde{\mathbf{f}}^{T}\boldsymbol{\beta} + b_m\frac{\eta_w}{w} = \boldsymbol{\beta}^{\mathrm{T}}\mathbf{b}\tilde{\mathbf{f}}^{\mathrm{T}}\boldsymbol{\beta} + \boldsymbol{\beta}^{\mathrm{T}}b_m\mathbf{B}\boldsymbol{\beta} - b_m[1 \quad 1 \quad \cdots \quad 1] \times \boldsymbol{\beta} = \boldsymbol{\beta}^{\mathrm{T}}\mathbf{b}\tilde{\mathbf{f}}^{\mathrm{T}}\boldsymbol{\beta} + b_m\{\boldsymbol{\beta}^{\mathrm{T}}\mathbf{B} - [1 \quad 1 \quad \cdots \quad 1]\}\boldsymbol{\beta}. \tag{37}$$

The expression in the curly brackets is denoted by ξ^{T}. Since $0 \leq \beta_i \leq 1$ it follows:

$$\min_j \frac{b_j}{b_i} - 1 \leq \xi_i \leq \max_j \frac{b_j}{b_i} - 1$$
$$\min_j \frac{b_j - b_i}{b_i} \leq \xi_i \leq \max_j \frac{b_j - b_i}{b_i} \tag{38}$$
$$|\xi^{\mathrm{T}}\boldsymbol{\beta}| \leq \frac{\max_{i,j} |b_j - b_i|}{\min_i |b_i|} < C_1$$

due to Assumption **A1** where C_1 is a constant. Error η_w can be deduced from (37)

$$\eta_w(t) = \frac{\boldsymbol{\beta}^{\mathrm{T}}(t)\mathbf{b} - b}{b_m}\tilde{\mathbf{f}}^{\mathrm{T}}(t)\boldsymbol{\beta}(t)w(t) + \xi^{\mathrm{T}}(t)\boldsymbol{\beta}(t)w(t) = f_w(t)w(t), \tag{39}$$

where $f_w(t)$ was introduced. Since $\boldsymbol{\beta}^{\mathrm{T}}\mathbf{b}$ (gain of the plant), $\xi^{\mathrm{T}}\boldsymbol{\beta}$ and $\tilde{\mathbf{f}}$ are bounded (see Assumption **A1** and Theorem 1), $|f_w|$ is always bounded, and it follows

$$|\eta_w(t)| \leq |w(t)| \sup_t |f_w| = \bar{f}_w |w(t)|. \tag{40}$$

If the gain of the controlled plant does not depend very much on the antecedent variables (elements of the vector $\mathbf{b}$ are similar), $(\boldsymbol{\beta}^{\mathrm{T}}\mathbf{b} - b)$ and $\xi^{\mathrm{T}}\boldsymbol{\beta}$ tend to zero and consecutively do $\bar{f}_w$ and η_w.

It follows from the second equation in (33)

$$\eta_y = \left[\frac{(\boldsymbol{\beta}^{\mathrm{T}}\mathbf{b})(\boldsymbol{\beta}^{\mathrm{T}}\mathbf{q})}{b_m} + \frac{(\boldsymbol{\beta}^{\mathrm{T}}\mathbf{a})}{b_m} - \frac{a_m}{b_m} - \frac{b\tilde{\mathbf{q}}^{\mathrm{T}}\boldsymbol{\beta}}{b_m}\right] y_p. \tag{41}$$

It will be shown below that the function in the brackets in (41) is bounded. Let us look at the first term in the brackets of (41)

$$\frac{(\boldsymbol{\beta}^{\mathrm{T}}\mathbf{b})(\boldsymbol{\beta}^{\mathrm{T}}\mathbf{q})}{b_m} = \frac{\boldsymbol{\beta}^{\mathrm{T}}\mathbf{b}\mathbf{q}^{*\mathrm{T}}\boldsymbol{\beta}}{b_m} + \frac{\boldsymbol{\beta}^{\mathrm{T}}\mathbf{b}\tilde{\mathbf{q}}^{\mathrm{T}}\boldsymbol{\beta}}{b_m} = \boldsymbol{\beta}^{\mathrm{T}}\left\{\frac{a_m}{b_m}\mathbf{B} - \frac{1}{b_m}\begin{bmatrix} a_1 & a_2\frac{b_1}{b_2} & \cdots & a_k\frac{b_1}{b_k} \\ a_1\frac{b_2}{b_1} & a_2 & \cdots & a_k\frac{b_2}{b_k} \\ \vdots & \vdots & \ddots & \\ a_1\frac{b_k}{b_1} & a_2\frac{b_k}{b_2} & & a_k \end{bmatrix}\right\} \times \boldsymbol{\beta} + \frac{\boldsymbol{\beta}^{\mathrm{T}}\mathbf{b}\tilde{\mathbf{q}}^{\mathrm{T}}\boldsymbol{\beta}}{b_m}. \tag{42}$$

The matrix in the brackets will be denoted by $\mathbf{A}$ in the following. Equation (41) can be rewritten as

$$\eta_y = \left[\boldsymbol{\beta}^{\mathrm{T}}\left(\frac{a_m}{b_m}\mathbf{B} - \frac{1}{b_m}\mathbf{A}\right)\boldsymbol{\beta} + \frac{\boldsymbol{\beta}^{\mathrm{T}}\mathbf{b}\tilde{\mathbf{q}}^{\mathrm{T}}\boldsymbol{\beta}}{b_m} + \frac{(\boldsymbol{\beta}^{\mathrm{T}}\mathbf{a})}{b_m} - \frac{a_m}{b_m} - \frac{b\tilde{\mathbf{q}}^{\mathrm{T}}\boldsymbol{\beta}}{b_m}\right] y_p$$
$$= \frac{a_m}{b_m}(\boldsymbol{\beta}^{\mathrm{T}}\mathbf{B} - [1 \quad 1 \quad \cdots \quad 1])\boldsymbol{\beta} y_p - \frac{1}{b_m}\{\boldsymbol{\beta}^{\mathrm{T}}\mathbf{A} - \mathbf{a}^{\mathrm{T}}\}\boldsymbol{\beta} y_p + \frac{\boldsymbol{\beta}^{\mathrm{T}}\mathbf{b} - b}{b_m}\tilde{\mathbf{q}}^{\mathrm{T}}\boldsymbol{\beta} y_p. \tag{43}$$

The expression in the parentheses in the first term is equal to ξ^{T}, while the expression in the curly brackets is denoted by χ^{T}. Since $0 \leq \beta_i \leq 1$, the i-th element of the vector χ can be bounded from above and below:

$$\min_j a_i\frac{b_j - b_i}{b_i} \leq \chi_i \leq \max_j a_i\frac{b_j - b_i}{b_i}$$
$$|\chi^{\mathrm{T}}\boldsymbol{\beta}| \leq \max_i a_i \frac{\max_{i,j}|b_i - b_j|}{\min_i b_i} < C_2 \tag{44}$$

due to Assumptions **A1** and **A2** where C_2 is a constant. Finally η_y can be expressed as:

$$\eta_y = \frac{a_m}{b_m}\,\xi^{\mathrm{T}}\boldsymbol{\beta} y_p - \frac{1}{b_m}\,\chi^{\mathrm{T}}\boldsymbol{\beta} y_p + \frac{\boldsymbol{\beta}^{\mathrm{T}}\mathbf{b} - b}{b_m}\,\tilde{\mathbf{q}}^{\mathrm{T}}\boldsymbol{\beta} y_p = f_y y_p. \tag{45}$$

It can be seen from (45) that the introduced f_y is a function of time. For our purposes only the upper bound on $|f_y(t)|$ is important. Since $\xi^{\mathrm{T}}\boldsymbol{\beta}$, $\chi^{\mathrm{T}}\boldsymbol{\beta}$, $\tilde{\mathbf{q}}$ (see Theorem 1) and $(\boldsymbol{\beta}^{\mathrm{T}}\mathbf{b} - b)/b_m$ are bounded, it follows

$$|\eta_y(t)| \le |y_p(t)| \sup_t |f_y(t)| = \bar{f}_y |y_p(t)|. \tag{46}$$

According to (45), $\bar{f}_y$ illustrates the nonlinearity of the plant (or better, its gain). If the elements of the vector $\mathbf{b}$ tend to a constant, $\bar{f}_y$ tends to 0. For linear plants or such that the gain of the plant is independent of $\boldsymbol{\beta}$, $\bar{f}_y$ is zero.

Final result follows from (28) and (33):

$$\begin{aligned}\dot{e} &= -a_m e + b_m\left(\frac{b}{b_m}\,\tilde{\mathbf{f}}^{\mathrm{T}}\boldsymbol{\beta} w - \frac{b}{b_m}\,\tilde{\mathbf{q}}^{\mathrm{T}}\boldsymbol{\beta} y_p + \eta_w - \eta_y\right)\\ &\quad + \Delta_u'(p)u - \Delta_y'(p)y_p + d'\\ &= -a_m e + b_m\left(\frac{b}{b_m}\,\tilde{\mathbf{f}}^{\mathrm{T}}\boldsymbol{\beta} w - \frac{b}{b_m}\,\tilde{\mathbf{q}}^{\mathrm{T}}\boldsymbol{\beta} y_p + f_w w - f_y y_p\right.\\ &\quad \left. + \Delta_u(p)u - \Delta_y(p)y_p + d\right),\end{aligned} \tag{47}$$

where the introduction of $\Delta_u(p)$ and $\Delta_y(p)$ is obvious. In the following the expression in the parentheses in (47) shall be substituted by η to simplify the notation, i.e.

$$\begin{aligned}\eta(t) &= f_w(t)w(t) - f_y(t)y_p(t) + \Delta_u(p)u(t)\\ &\quad - \Delta_y(p)y_p(t) + d(t).\end{aligned} \tag{48}$$

The expression in (47) is the so-called error model of the system that connects parameter vector errors with the tracking error.

3.2.3. *Boundedness and convergence of ε*

Theorem 2: *The adaptive law described by* (20), (23), *and* $m^2 = 1 + n_s^2$ *together with error model* (47) *guarantees:*

- $\varepsilon, \tilde{\mathbf{f}}, \tilde{\mathbf{q}} \in \mathcal{L}_\infty$;
- $\varepsilon, \varepsilon n_s, \varepsilon m \in \mathcal{S}\left(\dfrac{\eta^2}{m^2} + \nu_0^2\right)$; and
- $\dot{\mathbf{f}}, \dot{\mathbf{q}} \in \mathcal{S}\left(\dfrac{\eta^2}{m^2} + \nu_0^2\right)$,

if $\dfrac{\eta}{m} \in \mathcal{L}_\infty$.

The proof is given in appendix C.

3.2.4. *Boundedness of all the signals in the system and convergence of the tracking error*

It remains to be solved how to design normalizing variable m. Theorems 1 and 2 demand that

$$\frac{w}{m}, \frac{y_p}{m}, \frac{\eta}{m} \in \mathcal{L}_\infty.$$

According to (48) that defines η, we can propose the following formula:

$$\begin{aligned} m^2 &= 1 + n_s^2\\ n_s^2 &= w^2 + y_p^2 + m_s\\ \dot{m}_s &= -\delta_0 m_s + u^2 + y_p^2 \quad m_s(0) = 0,\end{aligned} \tag{49}$$

where $\delta_0 > 0$ and will be discussed below.

Theorem 3: *The model reference adaptive control system, described by* (18), (20), (23) *and* (49), *is globally stable, i.e. all the signals in the system are bounded and the tracking error has the following properties*

- $e \in \mathcal{L}_\infty$; *and*
- $e \in \mathcal{S}(\Delta_2^2 + \bar{d}^2 + \nu_0^2)$,

if the following conditions are satisfied:

- $\dfrac{c}{\alpha_0^2}\,\Delta_\infty^2 + \dfrac{c}{\alpha_0^2} + c\Delta_\infty^2 < 1$;
- $c\Delta_2^2 + c\nu_0^2 < \delta_0$;
- $\Delta_u(s)$, $\Delta_y(s)$ and $G_m(s)$ *are analytic in* $\mathrm{Re}[s] \ge -\dfrac{\delta_0}{2}$;
- *reference signal w is continuous; and*
- $\boldsymbol{\beta}$ *is a function of continuous signals,*

where

- $\Delta_\infty = \max(\|\Delta_u(s)\|_{\infty\delta_0}, \|\Delta_y(s)\|_{\infty\delta_0} + \bar{f}_y)$;
- $\Delta_2 = \max(\|\Delta_u(s)\|_{2\delta_0}, \|\Delta_y(s)\|_{2\delta_0}, \bar{f}_w, \bar{f}_y)$;
- $\bar{d} = \sup_t |d(t)|$;
- α_0 *is an arbitrary constant such that* $\alpha_0 > a_m$; *and*
- c *are constants that depend on different system parameters (reference model,* δ_0, ν_0, *and other).*

Furthermore, estimated control gains converge to the residual set:

$$\left\{ f_i, q_i \mid |f_i| < \frac{1}{\nu_0},\ |q_i| < \frac{1}{\nu_0},\ i = 1, \ldots, k \right\}. \tag{50}$$

The proof is given in appendix C.

Remark 1: In the theorem, transfer functions in the Laplace domain are used instead of the equivalent operators in the time domain. If the analyticity or norms of the operators are needed, the description in the s domain is more suitable. If the input–output relations of the system are used, the description in the time domain is usually used. Both notations are used interchangeably in the rest of the paper.

Remark 2: If the unmodelled dynamics of the plant that are represented by Δ_∞ are small enough, then by choosing α_0 (which is not a design parameter but is only used in the stability proof, therefore it is arbitrary) large enough, the first condition can always be satisfied. The first term in the second condition also gives information about the unmodelled dynamics. Together with the choice of the leakage parameter ν_0, the second condition represents the lower bound on δ_0 that would still assure stable behaviour. On the other hand, δ_0 is also limited from above with the third condition. The dominant limitation of the third condition is usually the condition on $G_m(s)$ since it is not advisable to choose the reference model 'quicker' than parasitic dynamics due to robustness issues. The continuity of the reference signal is not so stringent as it appears at first sight. We can see from (112) that by choosing α_0 large enough, arbitrary large derivatives of the reference signal are allowed. Since the adaptive control is usually realized by a digital controller, a reference signal that consists of square impulses can be treated as continuous with large derivatives in points of discontinuity.

Remark 3: The parameters f_i and q_i will converge to the residual set (50) if the adaptive error ε and the fulfilment of the corresponding membership functions β_i are non-zero. If these conditions are not satisfied, the parameters will be frozen. This means that the asymptotic convergence of the parameters is not guaranteed. On the other hand, the parameters that may be out of the bounds (50) do not contribute to the control signal since β_i is explicitly present in the control law.

Remark 4: The consequence of the second property of Theorem 3 is that short bursts of the signals are possible (they are quite usual in many forms of adaptive systems; e.g. Anderson 1985) but they are of finite amplitude and their duration is relatively short.

Remark 5: The convergence set (50) also represents potential danger in the case when the plant itself is unstable and no element of the set (50) provides stable behaviour of the plant. But this is a known problem of the adaptation with leakage as shown by Rey *et al.* (1989). To avoid it, a controller parameterization $(\hat{f}_1, \hat{f}_2, \ldots, \hat{f}_k, \hat{q}_1, \hat{q}_2, \ldots, \hat{q}_k)$ has to be known which assures stability of the system. Then a slightly modified adaptive law (20) should be used to obtain stable system:

$$\begin{aligned} \dot{f}_i &= -\gamma_{fi} b_{\text{sign}} \varepsilon w \beta_i - \gamma_{fi} |\varepsilon m| \nu_0 (f_i - \hat{f}_i) \beta_i \\ &\qquad\qquad i = 1, 2, \ldots, k \\ \dot{q}_i &= \gamma_{qi} b_{\text{sign}} \varepsilon y_p \beta_i - \gamma_{qi} |\varepsilon m| \nu_0 (q_i - \hat{q}_i) \beta_i \\ &\qquad\qquad i = 1, 2, \ldots, k. \end{aligned} \tag{51}$$

Remark 6: The problem of choosing the design parameters Γ, ν_o and δ_0 is still quite open. This is an everlasting problem in adaptive control. Some guidelines on choosing ν_o and δ_0 can be obtained from the conditions of Theorem 3. The latter does not impose any limitations on adaptive gain, but it is generally known that its choice is of crucial importance for the good performance of the adaptive system. As always, it turns out that any prior knowledge that is available to the designer can be used to improve the performance or the robustness of the over-all system.

4. Simulation example

A comparison between the proposed algorithm and the classical MRAC with e_1-modification will be given by testing them on a simulated model. A simulated plant was chosen since it was easier to make the same operating conditions than it would be when testing on a real plant. The model used here was the extended model of Rohrs *et al.* (1985) that can be rewritten in state space form:

$$\begin{aligned} \dot{y}_p &= -y_p + 2u_f \\ \dot{u}_f &= 229x_1 - 30u_f \\ \dot{x}_1 &= -u_f + u, \end{aligned} \tag{52}$$

where the part of the system between the plant input u and u_f represents the parasitic dynamics, while the first equation in (52) describes the nominal plant (the one used for control design). The plant was made nonlinear by adding extra terms to (52). Some properties of the original system were preserved, namely the linearized behaviour in the nominal operating point $(u = 0,\ y_p = 0)$ and the 'relative order' of the plant (meaning that u and x_1 do not influence $\dot{y}_p$ directly even in the form of higher powers). The resulting system used for simulations was:

$$\begin{aligned} \dot{y}_p &= -y_p + 2u_f + (-0.5y_p + 0.1u_f)^2 \\ &\quad + (-0.6y_p + 0.1u_f)^3 \\ \dot{u}_f &= 229x_1 - 30u_f + 6x_1^2 - 2x_1u_f - 0.1u_f^2 \\ \dot{x}_1 &= -u_f + u + 0.01u^2 - 0.01uu_f - 0.01u_f^2. \end{aligned} \tag{53}$$

By analysing the plant (53), it can be seen that it is highly nonlinear. Note that the parasitic dynamics are also nonlinear, not just the dominant part as was assumed when deriving the control algorithm. This means that this example will also test the ability of the proposed control to cope with nonlinear parasitic dynamics. The coefficients of the linearized system in different operating points depend on u, x_1, u_f and y_p, even though that only y_p will be used as a fuzzification variable, which is again a violation of the basic assumptions, but it still produces fairly good results.

The design objective was that the output of the plant follows the output of the reference model, i.e. the same as used by Rohrs *et al.* (1985): $3/(s+3)$. Three experiments were conducted, one with a PI controller, one with a classical MRAC and the last with DFMRAC. The reference signal was the same in all cases and it consisted of a periodic signal followed by two big steps (the last period and the steps are shown in figure 1). The PI controller (and not PID) was used because the modelled part of the plant and the reference model are of the first order and it is easy to design the PI controller to meet the design objective.

The model of the plant in the nominal operating point ($y_p = 0$) is $2/(s+1)$ yielding a PI controller with $K_p = 1.5$ and $T_i = 1$. The results are shown in figure 1. Since too slow a behaviour was obtained, the controller was designed again in a different operating point ($y_p = 4$ was chosen since the parameters of the plant are 'average' there), resulting in $K_p = 1.5$ and $T_i = 0.33$.

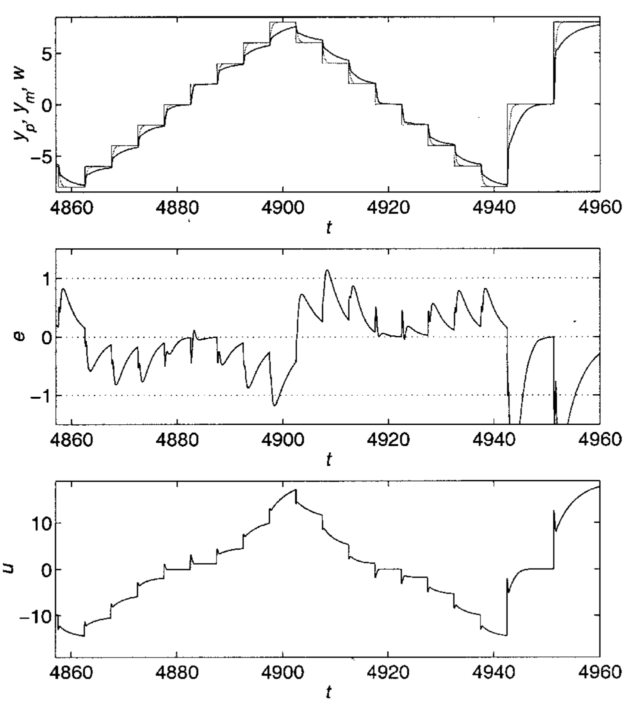

Figure 1. The PI controller designed in $y_p = 0$: time plots of the reference signal and outputs of the plant and the reference model (upper), time plot of tracking error (middle) and time plot of the control signal (lower).

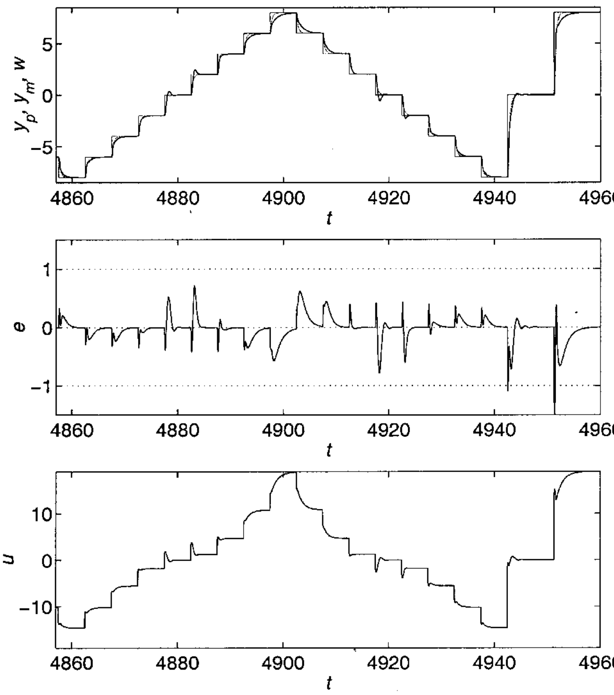

Figure 2. PI controller designed in $y_p = 4$: time plots of the reference signal and outputs of the plant and the reference model (upper), time plot of tracking error (middle) and time plot of the control signal (lower).

The results are shown in figure 2, where the system is underdamped around $y_p = 0$ and overdamped around $y_p = 8$.

In the case of DFMRAC, y_p was chosen as the fuzzification variable. Eleven triangular membership functions that were evenly distributed between -10 and 10 were selected. Our motivation for this choice was again that no prior knowledge on the nonlinearity was available to the designer. If some information was available, it could be used to improve performance. Since in the case of a linear plant DFMRAC becomes equivalent to a classical MRAC with e_1-modification, all design parameters are equivalent. Therefore, we used the same parameters in both experiments, namely $\gamma_f = \gamma_q = 2$, $\nu_0 = 0.1$ and $\delta_0 = 0.5$, to enable impartial comparison. Figures 3 and 4 show the results of the classical MRAC with e_1-modification: the former shows a period of system responses after the adaptation has settled, the latter depicts time plots of the estimated parameters. In figures 5 and 6, the same signals are shown for the proposed DFMRAC. Since **f** and **q** are vectors, all elements of the vectors are depicted.

The experiments show that the performance of the DFMRAC is much better than the performance of the other two approaches. Very good results are obtained in the case of DFMRAC even though the parasitic dynamics are nonlinear and linearized parameters

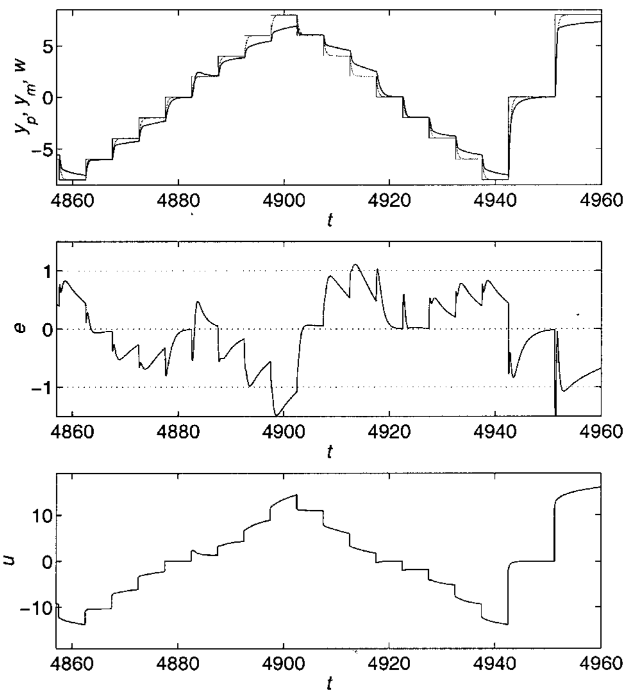

Figure 3. Classical MRAC with e_1-modification: time plots of the reference signal and outputs of the plant and the reference model (upper), time plot of tracking error (middle) and time plot of the control signal (lower).

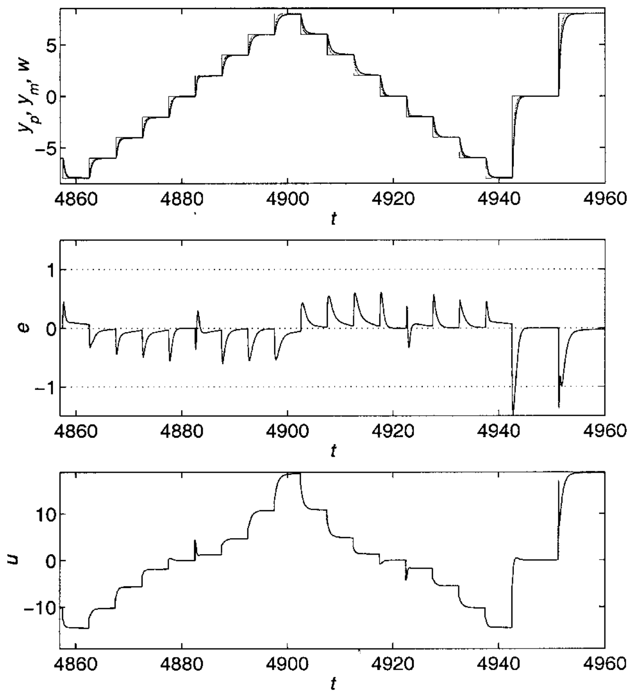

Figure 5. DFMRAC: time plots of the reference signal and outputs of the plant and the reference model (upper), time plot of tracking error (middle) and time plot of the control signal (lower).

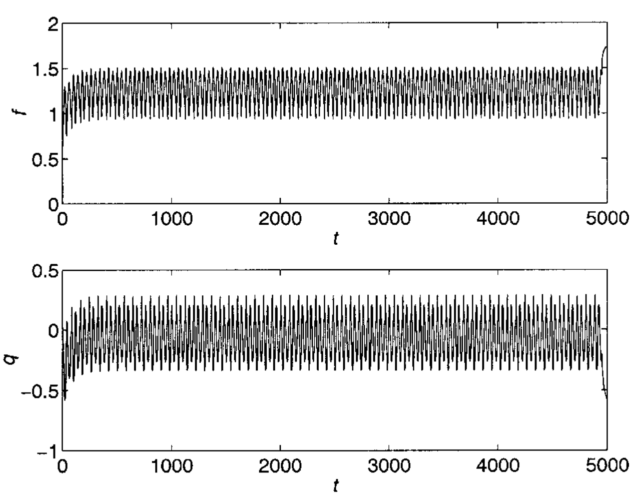

Figure 4. Classical MRAC with e_1-modification: time plots of feedforward (upper) and feedback (lower) control gains.

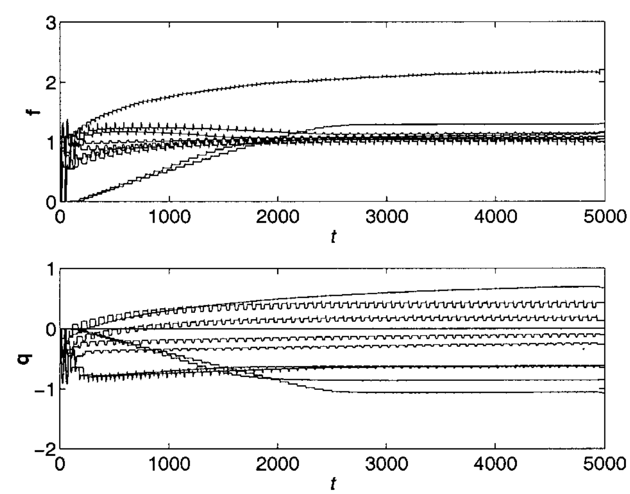

Figure 6. DFMRAC: time plots of feedforward (upper) and feedback (lower) control gains.

depend not only on the fuzzification variable, but also on others. The spikes in the middle figure in figure 4 are consequences of the fact that the plant of 'relative degree' 3 is forced to follow the reference model of relative degree 1. Since the plant is nonlinear a decision has to be made as to what operating point one should tune the PI controller. The consequences of the choice influence the performance of the system drastically.

The drawback of DFMRAC is relatively slow convergence since the parameters are only adapted when the corresponding membership is non-zero. This drawback can be overcome by using classical MRAC in the beginning when there are no parameter estimates or the estimates are bad. When the system approaches the desired behaviour, the adaptation can switch to that proposed by initializing all elements of vectors **f** and **q** by estimated scalar parameters f and q, respectively. In our case, all the estimates were 0 in the beginning, resulting in the fact that the controller gains were too small in the beginning. The output of the plant was also too small and some periods of the reference signal were needed so that the output reached membership functions around 6 and 8. This means that for some time the corresponding control gains were 0 since they were not adapted. The system was in the 'magic circle' that prevented it from reaching the desired behaviour in the beginning. Some experiments have shown that if the reference model output was chosen as a fuzzification variable, this start-up interval was shortened, which is understandable since adaptation started in all fuzzy domains in the first period. When the system was moving towards the desired behaviour, the difference between y_p and y_m as fuzzification variables did not make much difference. The problem was that the approach with y_m as fuzzification variable did not have any background in the fuzzy model.

5. Conclusions

A direct fuzzy adaptive control algorithm was presented here. It was shown in Theorem 3 that the closed-loop system is stable provided some conditions about the size of disturbances and high-order parasitics are met. The advantage of the proposed approach is that it is very simple to design, but it still offers the advantages of nonlinear and adaptive controllers. It was shown in the example that good results can be obtained if a third-order plant is treated as a first-order plant. It also proves very successful when disturbances are present only in certain operating regions since only estimates of the corresponding parameters are bad. When the system leaves those conditions (fuzzy domains), the perfect function of the controller is restored instantly. The drawback of the approach is a long time of adaptation which is the result of the large number of parameters that have to be estimated. To speed up the adaptation, classical adaptation can be used in the early phase, followed by fuzzy adaptation when the classical adaptation quasi-settles. Switching from the former to the latter is very easy and does not cause any bumps.

Appendix A

Some functional analysis preliminaries are given on norms and smallness of signals in a mean-square sense that are used frequently throughout the paper. They are given here for the sake of completeness. More complete treatment can be found in textbooks on functional analysis. A very good summary needed for use in control in general and especially in robust adaptive control is given in Ioannou and Sun (1996).

Since most of the signals analysed here do not have finite $\mathcal{L}_p$ norms, $\mathcal{L}_{pe}$ norms are used instead. They are defined as usual $\mathcal{L}_p$ norms but the upper limit of the integral is t instead of infinity. If a function has a finite $\mathcal{L}_{pe}$ norm we say it belongs to the $\mathcal{L}_{pe}$ set. For stability, analysis of the proposed algorithm exponentially weighted $\mathcal{L}_2$ norms was shown to be particularly useful. They are defined as

$$\|\mathbf{x}_t\|_{2\delta} \triangleq \left(\int_0^t e^{-\delta(t-\tau)}\mathbf{x}^{\mathrm{T}}(\tau)\mathbf{x}(\tau)\,\mathrm{d}\tau\right)^{1/2}, \tag{54}$$

where $\delta \geq 0$ is a constant. If the LTI system is given by

$$y = H(s)u, \tag{55}$$

where $H(s)$ is a proper rational function of s that is analytic in $\mathrm{Re}[s] \geq -\delta/2$ for some $\delta \geq 0$ and $u \in \mathcal{L}_{2e}$ then

$$\|y_t\|_{2\delta} \leq \|H(s)\|_{\infty\delta}\|u_t\|_{2\delta}, \tag{56}$$

where

$$\|H(s)\|_{\infty\delta} \triangleq \sup_{\omega}\left|H\left(j\omega - \frac{\delta}{2}\right)\right|. \tag{57}$$

Furthermore, when $H(s)$ is strictly proper, we have

$$|y(t)| \leq \|H(s)\|_{2\delta}\|u_t\|_{2\delta}, \tag{58}$$

where

$$\|H(s)\|_{2\delta} \triangleq \frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{\infty}\left|H\left(j\omega - \frac{\delta}{2}\right)\right|^2 \mathrm{d}\omega\right)^{1/2}. \tag{59}$$

Definition of smallness in mean square sense (Ioannou and Sun 1996). Let $\mathbf{x} : [0,\infty) \to \mathbb{R}^n, w : [0,\infty) \to \mathbb{R}^+$ where $\mathbf{x} \in \mathcal{L}_{2e}$, $w \in \mathcal{L}_{1e}$ and consider the set

$$\mathcal{S}(w) = \left\{\mathbf{x}, w \left| \int_t^{t+T} \mathbf{x}^{\mathrm{T}}(\tau)\mathbf{x}(\tau)\,\mathrm{d}\tau \right.\right.$$
$$\left. \leq c_0 \int_t^{t+T} w(\tau)\,\mathrm{d}\tau + c_1,\ \forall t, T \geq 0\right\}, \tag{60}$$

where $c_0, c_1 \geq 0$ are some finite constants. We say that $\mathbf{x}$ is w -small in the mean-square sense if $\mathbf{x} \in \mathcal{S}(w)$.

Appendix B

Some useful lemmas are given here. They are indispensable since they are used repeatedly in the proofs of the theorems. They are given here without explicit proofs. Most are proven implicitly; the others are very simple to prove, and therefore the proofs are omitted.

If $\mathbf{x}(t)$ is a vector and $\alpha(t)$ is a scalar or vice versa then

$$\|(\mathbf{x}\alpha)_t\|_{2\delta} = \left(\int_0^t e^{-\delta(t-\tau)}\mathbf{x}^{\mathrm{T}}(\tau)\mathbf{x}(\tau)\alpha^{\mathrm{T}}(\tau)\alpha(\tau)\,\mathrm{d}\tau\right)^{1/2}$$
$$\leq \left(\int_0^t e^{-\delta(t-\tau)}\mathbf{x}^{\mathrm{T}}(\tau)\mathbf{x}(\tau)\,\mathrm{d}\tau\right)^{1/2}$$
$$\times \sup_t (\alpha^{\mathrm{T}}(t)\alpha(t))^{1/2} = \|\mathbf{x}_t\|_{2\delta} \sup_t |\alpha(t)|. \tag{61}$$

If $\mathbf{x}(t)$ and $\alpha(t)$ are vectors then

$$\|(\mathbf{x}^{\mathrm{T}}\alpha)_t\|_{2\delta}$$
$$= \left(\int_0^t e^{-\delta(t-\tau)}\mathbf{x}^{\mathrm{T}}(\tau)\alpha(\tau)\alpha^{\mathrm{T}}(\tau)\mathbf{x}(\tau)\,\mathrm{d}\tau\right)^{1/2}$$
$$\leq \left(\int_0^t e^{-\delta(t-\tau)}\mathbf{x}^{\mathrm{T}}(\tau)\lambda_{\max}(\alpha(\tau)\alpha^{\mathrm{T}}(\tau))\mathbf{x}(\tau)\,\mathrm{d}\tau\right)^{1/2}$$
$$= \left(\int_0^t e^{-\delta(t-\tau)}\mathbf{x}^{\mathrm{T}}(\tau)\mathbf{x}(\tau)|\alpha(\tau)|^2\,\mathrm{d}\tau\right)^{1/2}$$
$$= \|(\mathbf{x}|\alpha|)_t\|_{2\delta}, \tag{62}$$

where $\lambda_{\max}(\mathbf{A})$ denotes the largest eigenvalue of the matrix $\mathbf{A}$. The only non-zero eigenvalue of the matrix $\mathbf{cc}^{\mathrm{T}}$ is $|\mathbf{c}|$ for any vector $\mathbf{c} \neq \mathbf{0}$. Combining (61) and (62) we get

$$\|(\mathbf{x}^{\mathrm{T}}\alpha)_t\|_{2\delta} \leq \|(\mathbf{x}|\alpha|)_t\|_{2\delta} \leq \|\mathbf{x}_t\|_{2\delta} \sup_t |\alpha(t)|. \tag{63}$$

If $\mathbf{x}(t)$ is a vector then the upper bound on $\|\mathbf{x}_t\|_{2\delta}$ is

$$\|\mathbf{x}\|_{2\delta} = \left(\int_0^t e^{-\delta(t-\tau)}\mathbf{x}^{\mathrm{T}}(\tau)\mathbf{x}(\tau)\,\mathrm{d}\tau\right)^{1/2}$$
$$\leq \left(\int_0^t e^{-\delta(t-\tau)}\,\mathrm{d}\tau\right)^{1/2} \sup_t (\mathbf{x}^{\mathrm{T}}(t)\mathbf{x}(t))^{1/2} \tag{64}$$
$$= \sqrt{\frac{1-e^{-\delta t}}{\delta}} \sup_t |\mathbf{x}(t)| < \frac{1}{\sqrt{\delta}} \sup_t |\mathbf{x}(t)|.$$

Since elements of vector $\boldsymbol{\beta}$ are normalized it follows

$$\sup_t |\boldsymbol{\beta}(t)| = \sup_t \sqrt{\sum_{i=1}^k \beta_i^2(t)} \leq \sqrt{\sum_{i=1}^k \beta_i} = 1. \tag{65}$$

If $f(t)$, $g(t)$ and $h(t)$ are scalar functions of time and $\mathbf{x}_i(t)$ are vector functions of time $(i = 1, 2, \ldots, k)$ then

- $\|(fh)_t\| + \|(gh)_t\|$
$$= \sqrt{\int_0^t e^{-\delta(t-\tau)}(f^2(\tau) + g^2(\tau))h^2(\tau)\,\mathrm{d}\tau} \tag{66}$$
$$= \|(h\sqrt{f^2+g^2})_t\|,$$

- $\left\| \begin{matrix} \mathbf{x}_1(t) \\ \mathbf{x}_2(t) \\ \vdots \\ \mathbf{x}_k(t) \end{matrix} \right\|$
$$= \sqrt{\int_0^t e^{-\delta(t-\tau)}(\mathbf{x}_1^{\mathrm{T}}(t)\mathbf{x}_1(t) + \cdots + \mathbf{x}_k^{\mathrm{T}}(t)\mathbf{x}_k(t))\,\mathrm{d}\tau}$$
$$= \|\mathbf{x}_1(t)\| + \cdots + \|\mathbf{x}_k(t)\|. \tag{67}$$

If x_i $(i = 1, 2, \ldots, n)$ are real numbers then

$$\left(\sum_{i=1}^n x_i\right)^2 \leq n \sum_{i=1}^n x_i^2. \tag{68}$$

Appendix C

The extensive proofs of Theorems 2 and 3 are given. Both follow the general lines of the similar proofs presented by Ioannou and Sun (1996). There are, of course, many peculiarities of fuzzy modelling that make our proofs quite different from the mentioned ones.

Proof of Theorem 2

According to the error model (47) the tracking error e is obtained by filtering parameter errors and unmodelled term by a reference model G_m

$$e = G_m(p)\left(\frac{b}{b_m}\tilde{\mathbf{f}}^{\mathrm{T}}\boldsymbol{\beta}w - \frac{b}{b_m}\tilde{\mathbf{q}}^{\mathrm{T}}\boldsymbol{\beta}y_p + \eta\right). \tag{69}$$

By combining (23) and (69) we get

$$\varepsilon = G_m\left(\frac{b}{b_m}\tilde{\mathbf{f}}^{\mathrm{T}}\boldsymbol{\beta}w - \frac{b}{b_m}\tilde{\mathbf{q}}^{\mathrm{T}}\boldsymbol{\beta}y_p + \eta - \varepsilon n_s^2\right). \tag{70}$$

A Lyapunov function is proposed

$$V = \tfrac{1}{2}\tilde{\mathbf{f}}^{\mathrm{T}}\boldsymbol{\Gamma}_f^{-1}\tilde{\mathbf{f}} + \tfrac{1}{2}\tilde{\mathbf{q}}^{\mathrm{T}}\boldsymbol{\Gamma}_q^{-1}\tilde{\mathbf{q}} + \frac{1}{2|b|}\varepsilon^2. \tag{71}$$

The derivative of the Lyapunov function (71) is

$$\dot{V} = \tilde{\mathbf{f}}^{\mathrm{T}}\boldsymbol{\Gamma}_f^{-1}\dot{\tilde{\mathbf{f}}} + \tilde{\mathbf{q}}^{\mathrm{T}}\boldsymbol{\Gamma}_q^{-1}\dot{\tilde{\mathbf{q}}} + \frac{1}{|b|}\varepsilon\dot{\varepsilon}. \tag{72}$$

Since $\dot{\tilde{\mathbf{f}}} = \dot{\mathbf{f}}^*$ and $\dot{\tilde{\mathbf{q}}} = \dot{\mathbf{q}}^*$ it follows from (72) using (21) and (70)

$$\begin{aligned}\dot V &= \tilde{\mathbf f}^{\mathrm T}\boldsymbol\Gamma_f^{-1}(-\boldsymbol\Gamma_f b_{\mathrm{sign}}\varepsilon w\boldsymbol\beta - \boldsymbol\Gamma_f|\varepsilon m|\nu_0\mathbf F\boldsymbol\beta)\\ &\quad + \tilde{\mathbf q}^{\mathrm T}\boldsymbol\Gamma_q^{-1}(\boldsymbol\Gamma_q b_{\mathrm{sign}}\varepsilon y_p\boldsymbol\beta - \boldsymbol\Gamma_q|\varepsilon m|\nu_0\mathbf Q\boldsymbol\beta)\\ &\quad + \frac{1}{|b|}\varepsilon\Bigg(-a_m\varepsilon + b_m\bigg(\frac{b}{b_m}\tilde{\mathbf f}^{\mathrm T}\boldsymbol\beta w - \frac{b}{b_m}\tilde{\mathbf q}^{\mathrm T}\boldsymbol\beta y_p\\ &\quad - \varepsilon n_s^2 + \eta\bigg)\Bigg)\\ &= -\tilde{\mathbf f}^{\mathrm T} b_{\mathrm{sign}}\varepsilon w\boldsymbol\beta - \tilde{\mathbf f}^{\mathrm T}|\varepsilon m|\nu_0\mathbf F\boldsymbol\beta\\ &\quad + \tilde{\mathbf q}^{\mathrm T} b_{\mathrm{sign}}\varepsilon y_p\boldsymbol\beta - \tilde{\mathbf q}^{\mathrm T}|\varepsilon m|\nu_0\mathbf Q\boldsymbol\beta\\ &\quad - \frac{a_m}{|b|}\varepsilon^2 + \mathrm{sgn}(b)\tilde{\mathbf f}^{\mathrm T}\boldsymbol\beta w\varepsilon - \mathrm{sgn}(b)\tilde{\mathbf q}^{\mathrm T}\boldsymbol\beta y_p\varepsilon\\ &\quad + \frac{b_m}{|b|}(-\varepsilon^2 n_s^2 + \varepsilon\eta)\\ &= -|\varepsilon m|\nu_0(\tilde{\mathbf f}^{\mathrm T}\mathbf F + \tilde{\mathbf q}^{\mathrm T}\mathbf Q)\boldsymbol\beta - \frac{a_m}{|b|}\varepsilon^2\\ &\quad + \frac{b_m}{|b|}(-\varepsilon^2 n_s^2 + \varepsilon\eta).\end{aligned} \tag{73}$$

The last equality follows from the assumption **A3** that all b_i's and b have the same sign, i.e. b_{sign}.

What can be said about $-(\tilde{\mathbf f}^{\mathrm T}\mathbf F + \tilde{\mathbf q}^{\mathrm T}\mathbf Q)\boldsymbol\beta$?

$$\begin{aligned} -(\tilde{\mathbf f}^{\mathrm T}\mathbf F + \tilde{\mathbf q}^{\mathrm T}\mathbf Q)\boldsymbol\beta &= -\sum_{i=1}^{k}(\tilde f_i f_i\beta_i + \tilde q_i q_i\beta_i)\\ &= -\sum_{i=1}^{k}(\tilde f_i(f_i^* + \tilde f_i)\beta_i + \tilde q_i(q_i^* + \tilde q_i)\beta_i)\\ &= \sum_{i=1}^{k}(-\tilde f_i^2\beta_i - \tilde q_i^2\beta_i - f_i^*\tilde f_i\beta_i - q_i^*\tilde q_i\beta_i)\\ &\le \sum_{i=1}^{k}(-\tilde f_i^2\beta_i - \tilde q_i^2\beta_i + |f_i^*|\cdot|\tilde f_i|\beta_i + |q_i^*|\cdot|\tilde q_i|\beta_i)\\ &\le \sum_{i=1}^{k}(-\tilde f_i^2\beta_i - \tilde q_i^2\beta_i + |f_i^*|\cdot|\tilde f_i|\beta_i + |q_i^*|\cdot|\tilde q_i|\beta_i\\ &\quad + \frac{\beta_i}{2}(|\tilde f_i| - |f_i^*|)^2 + \frac{\beta_i}{2}(|\tilde q_i| - |q_i^*|)^2)\\ &= \sum_{i=1}^{k}\left(-\frac{\beta_i}{2}\tilde f_i^2 - \frac{\beta_i}{2}\tilde q_i^2 + \frac{\beta_i}{2}f_i^{*2} + \frac{\beta_i}{2}q_i^{*2}\right)\\ &= \sum_{i=1}^{k}\left(\frac{\beta_i}{2}(-\tilde f_i^2 - \tilde q_i^2 + f_i^{*2} + q_i^{*2})\right)\\ &\le -\min\left(\sum_{i=1}^{k}\left(\frac{\beta_i}{2}(\tilde f_i^2 + \tilde q_i^2)\right)\right)\\ &\quad + \max\left(\sum_{i=1}^{k}\left(\frac{\beta_i}{2}(f_i^{*2} + q_i^{*2})\right)\right)\\ &= 0 + \max_i\left(\tfrac12(f_i^{*2} + q_i^{*2})\right).\end{aligned} \tag{74}$$

The calculated upper bound of $-(\tilde{\mathbf f}^{\mathrm T}\mathbf F + \tilde{\mathbf q}^{\mathrm T}\mathbf Q)\boldsymbol\beta$ in (74) will be denoted by $\boldsymbol\theta^{*2}$. Using (74) and the inequality

$$\begin{aligned} -\frac{a_m}{|b|}\varepsilon^2 - \frac{b_m}{|b|}\varepsilon^2 n_s^2 &\le -\frac{\min(a_m, b_m)}{|b|}\varepsilon^2(1 + n_s^2)\\ &= -\frac{\min(a_m, b_m)}{|b|}\varepsilon^2 m^2\end{aligned}$$

it follows from (73):

$$\begin{aligned}\dot V &\le |\varepsilon m|\nu_0\boldsymbol\theta^{*2} - \frac{\min(a_m, b_m)}{|b|}\varepsilon^2 m^2 + \frac{b_m}{|b|}\varepsilon\eta\\ &\le |\varepsilon m|\left(\nu_0\boldsymbol\theta^{*2} - \frac{\min(a_m, b_m)}{|b|}|\varepsilon m| + \frac{b_m}{|b|}\frac{|\eta|}{m}\right).\end{aligned} \tag{75}$$

Since the desired control parameters (f_i^* and q_i^*) are finite, so is the constant $\boldsymbol\theta^{*2}$. The last term in the inequality (75) is bounded by assumption of the theorem. The derivative of the Lyapunov function will be definitely non-positive if

$$|\varepsilon m| > \frac{\nu_0|b|}{\min(a_m, b_m)}\boldsymbol\theta^{*2} + \frac{b_m}{\min(a_m, b_m)}\frac{|\eta|}{m}. \tag{76}$$

Since $|\varepsilon m|$ is positive if inequality (76) holds, $\dot V$ in (75) is strictly negative, not just non-positive when condition (76) is satisfied. Because $m \ge 1$ by construction it follows $|\varepsilon| \le |\varepsilon m|$ and large enough $|\varepsilon|$ causes that Lyapunov function starts decreasing. It was shown previously (see Theorem 1) that $\tilde{\mathbf f}$ and $\tilde{\mathbf q}$ are bounded. From these two facts it follows:

$$V, \varepsilon, \tilde{\mathbf f}, \tilde{\mathbf q} \in \mathcal L_\infty. \tag{77}$$

Inequality (75) can be rewritten as:

$$\begin{aligned}\dot V &\le -k_1^2\varepsilon^2 m^2 + \nu_0|\varepsilon m|\boldsymbol\theta^{*2} + k_2^2|\varepsilon m|\frac{|\eta|}{m}\\ &\le -k_1^2\varepsilon^2 m^2 + \nu_0|\varepsilon m|\boldsymbol\theta^{*2} + k_2^2|\varepsilon m|\frac{|\eta|}{m}\\ &\quad + \frac12\left(k_1|\varepsilon m| - \frac{1}{k_1}\left(k_2^2\frac{|\eta|}{m} + \nu_0\boldsymbol\theta^{*2}\right)\right)^2\\ &= -\frac{k_1^2}{2}\varepsilon^2 m^2 + \frac{1}{2k_1^2}\left(k_2^2\frac{|\eta|}{m} + \nu_0\boldsymbol\theta^{*2}\right)^2\\ &\le -\frac{k_1^2}{2}\varepsilon^2 m^2 + \frac{1}{2k_1^2}\left(k_2^2\frac{|\eta|}{m} + \nu_0\boldsymbol\theta^{*2}\right)^2\end{aligned}$$

$$+\frac{1}{2k_1^2}\left(k_2^2\frac{|\eta|}{m}-\nu_0\boldsymbol{\theta}^{*2}\right)^2=-\frac{k_1^2}{2}\varepsilon^2m^2$$
$$+\frac{k_2^4}{k_1^2}\frac{|\eta|^2}{m^2}+\frac{1}{k_1^2}\nu_0^2\boldsymbol{\theta}^{*4}, \quad (78)$$

where

$$\frac{\min(a_m, b_m)}{|b|}$$

was substituted by k_1^2 and

$$\frac{b_m}{|b|}$$

by k_2^2. Integrating both sides of the inequality (78), we obtain

$$\int_{t_0}^{t}\frac{k_1^2}{2}\varepsilon^2m^2\,\mathrm{d}\tau\le\int_{t_0}^{t}\left(\frac{k_2^4}{k_1^2}\frac{\eta^2}{m^2}+\frac{1}{k_1^2}\nu_0^2\boldsymbol{\theta}^{*4}\right)\mathrm{d}\tau$$
$$+V(t_0)-V(t) \quad (79)$$

for $\forall t\ge t_0$ and any $t_0\ge 0$. Because $V\in\mathcal{L}_\infty$ and $m^2=1+n_s^2$, it follows

$$\varepsilon, \varepsilon n_s, \varepsilon m\in\mathcal{S}\left(\frac{\eta^2}{m^2}+\nu_0^2\right), \quad (80)$$

where $\mathcal{S}(\cdot)$ gives information about the mean-square of the signals and is defined in appendix A.

From (21) it follows:

$$\begin{aligned}\dot{\mathbf{f}}&=-\boldsymbol{\Gamma}_f\varepsilon m\left(b_{\text{sign}}\frac{w}{m}+\operatorname{sgm}(\varepsilon m)\nu_0\mathbf{F}\right)\boldsymbol{\beta}\\ \dot{\mathbf{q}}&=-\boldsymbol{\Gamma}_q\varepsilon m\left(-b_{\text{sign}}\frac{y_p}{m}+\operatorname{sgn}(\varepsilon m)\nu_0\mathbf{Q}\right)\boldsymbol{\beta}\end{aligned} \quad (81)$$

and consecutively

$$\begin{aligned}|\dot{\mathbf{f}}|&\le c|\varepsilon m| \text{ since } \frac{w}{m},\mathbf{F}\in\mathcal{L}_\infty\\ |\dot{\mathbf{q}}|&\le c|\varepsilon m| \text{ since } \frac{y_p}{m},\mathbf{Q}\in\mathcal{L}_\infty.\end{aligned} \quad (82)$$

Combining (80) and (82) it follows

$$\dot{\mathbf{f}},\dot{\mathbf{q}}\in\mathcal{S}\left(\frac{\eta^2}{m^2}+\nu_0^2\right). \qquad \square$$

The proof of Theorem 3

In the following $\|(\cdot)\|$ denotes the $\mathcal{L}_{2\delta_0}$ norm, i.e. $\|(\cdot)_t\|_{2\delta_0}$.

By defining $\tilde{\boldsymbol{\theta}}^{\mathrm{T}}\triangleq[\tilde{\mathbf{f}}^{\mathrm{T}}\quad\tilde{\mathbf{q}}^{\mathrm{T}}]$ and $\omega^{\mathrm{T}}\triangleq[\boldsymbol{\beta}^{\mathrm{T}}w\quad-\boldsymbol{\beta}^{\mathrm{T}}y_p]$, (47) can be rewritten as

$$e=G_m(p)\left(\frac{b}{b_m}\tilde{\boldsymbol{\theta}}^{\mathrm{T}}\omega+\eta\right). \quad (83)$$

The normalizing signal m in (49) is equal to

$$m^2=1+w^2+y_p^2+\|u\|^2+\|y_p\|^2. \quad (84)$$

It will be shown that

$$\frac{\eta}{m},\frac{\|\eta\|}{m},\frac{u}{m},\frac{\|u\|}{m},\frac{y_p}{m},\frac{\|y_p\|}{m},\frac{\omega}{m},\frac{\|\omega\|}{m},\frac{\|\dot{y}_p\|}{m}\in\mathcal{L}_\infty.$$

If additionally $\dot{w}\in\mathcal{L}_\infty$ then

$$\frac{\|\dot{w}\|}{m}\in\mathcal{L}_\infty.$$

It follows from (48) by using property (56):

$$\begin{aligned}\|\eta\|&\le\|\Delta_u\|_{\infty\delta_0}\|u\|+\|\Delta_y\|_{\infty\delta_0}\|y_p\|\\&\quad+\bar{f}_w\|w\|+\bar{f}_y\|y_p\|+\|d\|\\&\le\frac{1}{\sqrt{\delta}}(\bar{f}_w\bar{w}+\bar{d})+\|\Delta_u\|_{\infty\delta_0}\|u\|\\&\quad+(\|\Delta_y\|_{\infty\delta_0}+\bar{f}_y)\|y_p\|\\&\le\frac{1}{\sqrt{\delta}}(\bar{f}_w\bar{w}+\bar{d})+\Delta_\infty m,\end{aligned} \quad (85)$$

where $\Delta_\infty=\max(\|\Delta_u\|_{\infty\delta_0},\|\Delta_y\|_{\infty\delta_0}+\bar{f}_y)$. Similarly it follows from (48), (58) and (68):

$$\begin{aligned}|\eta|&\le\|\Delta_u(s)\|_{2\delta_0}\|u_t\|_{2\delta_0}+\|\Delta_y(s)\|_{2\delta_0}\|(y_p)_t\|_{2\delta_0}\\&\quad+\bar{f}_w|w|+\bar{f}_y|y_p|+\bar{d}\\&\le\max(\|\Delta_u(s)\|_{2\delta_0},\|\Delta_y(s)\|_{2\delta_0},\bar{f}_w,\bar{f}_y)\\&\quad\times(\|u_t\|_{2\delta_0}+\|(y_p)_t\|_{2\delta_0}+|w|+|y_p|)+\bar{d}\\&\le2\max(\|\Delta_u(s)\|_{2\delta_0},\|\Delta_y(s)\|_{2\delta_0},\bar{f}_w,\bar{f}_y)m+\bar{d}\\&=2\Delta_2m+\bar{d},\end{aligned} \quad (86)$$

where $\Delta_2=\max(\|\Delta_u(s)\|_{2\delta_0},\|\Delta_y(s)\|_{2\delta_0},\bar{f}_w,\bar{f}_y)$. From (86) and (68) we have

$$\eta^2\le8\Delta_2^2m^2+2\bar{d}^2. \quad (87)$$

The boundedness of

$$\frac{\|u\|}{m},\quad\frac{\|y_p\|}{m}\quad\text{and}\quad\frac{y_p}{m}$$

follows directly from (84). Using (67), (61), (65), (64) and (84) we get

$$\|\omega\|=\|-\boldsymbol{\beta}w\|+\|-\boldsymbol{\beta}y_p\|\le\|w\|+\|y_p\|\le c\bar{w}+m. \quad (88)$$

Similarly:

$$\begin{aligned}|\boldsymbol{\omega}|&=\sqrt{|\boldsymbol{\beta}w|^2+|-\boldsymbol{\beta}y_p|^2}=\sqrt{|\boldsymbol{\beta}|^2w^2+|\boldsymbol{\beta}|^2y_p^2}\\&\le\sqrt{w^2+y_p^2}<m.\end{aligned} \quad (89)$$

The input signal u is calculated according to the formula

$$u=\boldsymbol{\theta}^{\mathrm{T}}\omega. \quad (90)$$

Owing to the boundedness of the parameter vector $\boldsymbol{\theta}$ that is guaranteed by the adaptive law (Theorem 1) and (89) it can be concluded that $u \le cm$.

Using (83) output of the plant can be written as

$$y_p = G_m(p)\left(w + \frac{b}{b_m}\tilde{\boldsymbol{\theta}}^{\mathrm{T}}\omega + \eta\right). \tag{91}$$

The consequence of (91) is

$$\dot{y}_p = pG_m(p)\left(w + \frac{b}{b_m}\tilde{\boldsymbol{\theta}}^{\mathrm{T}}\omega + \eta\right). \tag{92}$$

From (92) it follows by using (56), (64) and (85)

$$\begin{aligned}\|\dot{y}_p\| &\le \|sG_m(s)\|_{\infty\delta_0}\left(\|w\| + \frac{b}{b_m}\|\tilde{\boldsymbol{\theta}}^{\mathrm{T}}\omega\| + \|\eta\|\right)\\ &\le c\bar{w} + c\|\omega\| + c\|\eta\|\\ &\le c\bar{w} + c\bar{f}_w\bar{w} + c\bar{d} + c\Delta_\infty m + cm.\end{aligned} \tag{93}$$

The upper bound for the norm of the vector $\dot{\omega}$ is calculated from the norms of its elements (see equation 67)

$$\begin{aligned}\|\dot{\omega}\| &= \left\|\begin{matrix}\frac{d}{dt}(\boldsymbol{\beta}w)\\ \frac{d}{dt}(-\boldsymbol{\beta}y_p)\end{matrix}\right\| = \|\dot{\boldsymbol{\beta}}w + \boldsymbol{\beta}\dot{w}\| + \|-\boldsymbol{\beta}\dot{y}_p - \dot{\boldsymbol{\beta}}y_p\|\\ &\le \|\dot{\boldsymbol{\beta}}w\| + \|\boldsymbol{\beta}\dot{w}\| + \|\boldsymbol{\beta}\dot{y}_p\| + \|\dot{\boldsymbol{\beta}}y_p\|\\ &\le \sup_t|\dot{\boldsymbol{\beta}}|\left(\frac{\bar{w}}{\sqrt{\delta}} + \|y_p\|\right) + \sup_t|\boldsymbol{\beta}|\left(\frac{\bar{\dot{w}}}{\sqrt{\delta}} + \|\dot{y}_p\|\right)\\ &\le c\bar{w} + c\bar{f}_w\bar{w} + c\bar{\dot{w}} + c\bar{d} + c\Delta_\infty m + cm,\end{aligned} \tag{94}$$

where $\bar{\dot{w}} = \sup_t|\dot{w}(t)|$, i.e. reference signal $w(t)$ has to be continuous. When the membership functions depend only on the signals that are continuous (e.g. y_p and w when the above assumption holds), β_i, $i = 1, 2, \ldots, k$, are also continuous and their derivatives are finite all the time, so the last inequality in (94) finally follows.

It follows from (91) and (85)

$$\begin{aligned}\|y_p\| &= \|G_m(s)\|_{\infty\delta_0}\left(\|w\| + \frac{b}{b_m}\|\tilde{\boldsymbol{\theta}}^{\mathrm{T}}\omega\| + \|\eta\|\right)\\ &\le c\bar{w} + c\|\tilde{\boldsymbol{\theta}}^{\mathrm{T}}\omega\| + c\bar{f}_w\bar{w} + c\bar{d} + c\Delta_\infty m\end{aligned} \tag{95}$$

and

$$\begin{aligned}|y_p| &= \|G_m(s)\|_{2\delta_0}\left(\|w\| + \frac{b}{b_m}\|\tilde{\boldsymbol{\theta}}^{\mathrm{T}}w\| + \|\eta\|\right)\\ &\le c\bar{w} + c\|\tilde{\boldsymbol{\theta}}^{\mathrm{T}}\omega\| + c\bar{f}_w\bar{w} + c\bar{d} + c\Delta_\infty m.\end{aligned} \tag{96}$$

From (11) it follows:

$$\begin{aligned}u &= \frac{1}{\boldsymbol{\beta}^{\mathrm{T}}\mathbf{b}}(\dot{y}_p + (\boldsymbol{\beta}^{\mathrm{T}}\mathbf{a})y_p + \Delta_y'(p)y_p - \Delta_u'(p)u - d')\\ &= \left(\frac{1}{\boldsymbol{\beta}^{\mathrm{T}}\mathbf{b}}pG_m(p) + \frac{\boldsymbol{\beta}^{\mathrm{T}}\mathbf{a}}{\boldsymbol{\beta}^{\mathrm{T}}\mathbf{b}}G_m(p)\right)\left(w + \frac{b}{b_m}\tilde{\boldsymbol{\theta}}^{\mathrm{T}}\omega + \eta\right)\\ &\quad + \frac{b_m}{\boldsymbol{\beta}^{\mathrm{T}}\mathbf{b}}(\Delta_y(p)y_p - \Delta_u(p)u - d)\end{aligned} \tag{97}$$

and further:

$$\begin{aligned}\|u\| &\le \left(\sup_t\left|\frac{1}{\boldsymbol{\beta}^{\mathrm{T}}(t)\mathbf{b}}\right|\|sG_m(s)\|_{\infty\delta_0}\right.\\ &\quad\left. + \sup_t\left|\frac{\boldsymbol{\beta}^{\mathrm{T}}(t)\mathbf{a}}{\boldsymbol{\beta}^{\mathrm{T}}(t)\mathbf{b}}\right|\|G_m(s)\|_{\infty\delta_0}\right)\\ &\quad\times\left(\|w\| + \frac{b}{b_m}\|\tilde{\boldsymbol{\theta}}^{\mathrm{T}}\omega\| + \|\eta\|\right)\\ &\quad + \sup_t\left|\frac{b_m}{\boldsymbol{\beta}^{\mathrm{T}}(t)\mathbf{b}}\right|(\|\Delta_y(s)\|_{\infty\delta_0}\|y_p\|\\ &\quad + \|\Delta_u(s)\|_{\infty\delta_0}\|u\| + \|d\|)\\ &\le c\bar{w} + c\bar{f}_w\bar{w} + c\bar{d} + c\Delta_\infty m + c\|\tilde{\boldsymbol{\theta}}^{\mathrm{T}}\omega\|.\end{aligned} \tag{98}$$

Combining (95), (96) and (98) and using (68) the following inequality is obtained:

$$\begin{aligned}m^2 &= 1 + w^2 + y_p^2 + \|u\|^2 + \|y_p\|^2\\ &\le 1 + c\bar{w}^2 + c\bar{f}_w^2\bar{w}^2 + c\bar{d}^2 + c\Delta_\infty^2 m^2 + c\|\tilde{\boldsymbol{\theta}}^{\mathrm{T}}\omega\|^2.\end{aligned} \tag{99}$$

From (70) the error ε can be rewritten as

$$\varepsilon = G_m\left(\frac{b}{b_m}\tilde{\boldsymbol{\theta}}^{\mathrm{T}}\omega - \varepsilon n_s^2 + \eta\right). \tag{100}$$

The product $\tilde{\boldsymbol{\theta}}^{\mathrm{T}}\omega$ can be decomposed into

$$\tilde{\boldsymbol{\theta}}^{\mathrm{T}}\omega = \frac{1}{p + \alpha_0}(\tilde{\boldsymbol{\theta}}^{\mathrm{T}}\dot{\omega} + \dot{\tilde{\boldsymbol{\theta}}}^{\mathrm{T}}\omega) + \frac{\alpha_0}{p + \alpha_0}\tilde{\boldsymbol{\theta}}^{\mathrm{T}}\omega \tag{101}$$

where α_0 is an arbitrary positive number.

We can use (100) and the fact that

$$G_m(s) = \frac{b_m}{s + a_m}$$

further to derive from (101):

$$\begin{aligned}\tilde{\boldsymbol{\theta}}^{\mathrm{T}}\omega &= \frac{1}{p + \alpha_0}(\tilde{\boldsymbol{\theta}}^{\mathrm{T}}\dot{\omega} + \dot{\tilde{\boldsymbol{\theta}}}^{\mathrm{T}}\omega)\\ &\quad + \frac{\alpha_0(p + a_m)}{(p + \alpha_0)b}\varepsilon - \frac{\alpha_0 b_m}{(p + \alpha_0)b}\eta + \frac{\alpha_0 b_m}{(p + \alpha_0)b}\varepsilon n_s^2.\end{aligned} \tag{102}$$

The δ-shifted norms H_∞ of the transfer functions

$$\frac{1}{s+\alpha_0} \quad \text{and} \quad \frac{s+a_m}{s+\alpha_0} \quad \text{are} \quad \frac{1}{\alpha_0-\delta/2}$$

and 1, respectively. Since $\alpha_0 > a_m > \delta/2 > 0$ it follows:

$$\frac{1}{\alpha_0-\delta/2} < \frac{c}{\alpha_0}. \tag{103}$$

Using this the following inequality is obtained

$$\|\tilde{\boldsymbol{\theta}}^{\mathrm{T}}\omega\| \leq \frac{c}{\alpha_0}(\|\tilde{\boldsymbol{\theta}}^{\mathrm{T}}\dot{\omega}\| + \|\dot{\tilde{\boldsymbol{\theta}}}^{\mathrm{T}}\omega\|) + c\alpha_0\|\varepsilon\| + c\|\eta\| + c\|\varepsilon n_s^2\|. \tag{104}$$

Using (62) and(89) we get

$$\|\dot{\tilde{\boldsymbol{\theta}}}^{\mathrm{T}}\omega\| \leq \|\dot{\tilde{\boldsymbol{\theta}}}|\omega|\| \leq \|\dot{\tilde{\boldsymbol{\theta}}}m\|. \tag{105}$$

From (63) and (94) it follows

$$\|\tilde{\boldsymbol{\theta}}^{\mathrm{T}}\dot{\omega}\| \leq \|\dot{\omega}\| \sup_t |\tilde{\boldsymbol{\theta}}| \leq c\bar{w} + c\bar{f}_w\bar{w} + c\bar{\dot{w}} + c\bar{d} + c\Delta_\infty m + cm. \tag{106}$$

By inserting (105), (106) and (85) into (104) we get

$$\|\tilde{\boldsymbol{\theta}}^{\mathrm{T}}\omega\| \leq \frac{c}{\alpha_0}\|\dot{\tilde{\boldsymbol{\theta}}}m\| + c\alpha_0\|\varepsilon\| + c\|\varepsilon n_s^2\| + \frac{c}{\alpha_0}\bar{w} + \frac{c}{\alpha_0}\bar{f}_w\bar{w} + \frac{c}{\alpha_0}\bar{\dot{w}} + \frac{c}{\alpha_0}\bar{d} + \frac{c}{\alpha_0}\Delta_\infty m + \frac{c}{\alpha_0}m + c\bar{f}_w\bar{w} + c\bar{d} + c\Delta_\infty m. \tag{107}$$

Since ε is bounded (which is guaranteed by the adaptive law as shown before) and $n_s < m$ it follows

$$\|\tilde{\boldsymbol{\theta}}^{\mathrm{T}}\omega\| \leq \frac{c}{\alpha_0}\|\dot{\tilde{\boldsymbol{\theta}}}m\| + c\|\varepsilon n_s m\| + \left(\frac{c}{\alpha_0}\Delta_\infty + \frac{c}{\alpha_0} + c\Delta_\infty\right)m + \left(c\alpha_0\bar{\varepsilon} + \frac{c}{\alpha_0}\bar{w} + \frac{c}{\alpha_0}\bar{f}_w\bar{w} + \frac{c}{\alpha_0}\bar{\dot{w}} + \frac{c}{\alpha_0}\bar{d} + c\bar{f}_w\bar{w} + c\bar{d}\right). \tag{108}$$

Using (66) we get

$$\frac{c}{\alpha_0}\|\dot{\tilde{\boldsymbol{\theta}}}m\| + c\|\varepsilon n_s m\| = \frac{c}{\alpha_0}\||\dot{\tilde{\boldsymbol{\theta}}}|m\| + c\|\varepsilon n_s m\| \leq c\|gm\| \tag{109}$$

where

$$g^2 = \frac{|\dot{\tilde{\boldsymbol{\theta}}}|^2}{\alpha_0^2} + (\varepsilon n_s)^2.$$

Since

$$\varepsilon n_s, \dot{\tilde{\boldsymbol{\theta}}} \in \mathcal{S}\left(\frac{\eta^2}{m^2} + \nu_0^2\right)$$

it also holds that

$$g \in \mathcal{S}\left(\frac{\eta^2}{m^2} + \nu_0^2\right)$$

or by using (87)

$$g \in \mathcal{S}\left(\Delta_2^2 + \frac{\bar{d}^2}{m^2} + \nu_0^2\right). \tag{110}$$

If the term in the parentheses in (108) is denoted by c', the inequality (108) becomes

$$\|\tilde{\boldsymbol{\theta}}^{\mathrm{T}}\omega\| \leq c\|gm\| + \left(\frac{c}{\alpha_0}\Delta_\infty + \frac{c}{\alpha_0} + c\Delta_\infty\right)m + c'. \tag{111}$$

By using (99) and (111) it follows

$$\begin{aligned} m^2 &\leq 1 + c\bar{w}^2 + c\bar{f}_w^2\bar{w}^2 + c\bar{d}^2 + c\Delta_\infty^2 m^2 + c\|gm\|^2 + \left(\frac{c}{\alpha_0^2}\Delta_\infty^2 + \frac{c}{\alpha_0^2} + c\Delta_\infty^2\right)m^2 + cc'^2 \\ &\leq c\|gm\|^2 + \left(\frac{c}{\alpha_0^2}\Delta_\infty^2 + \frac{c}{\alpha_0^2} + c\Delta_\infty^2\right)m^2 + \left(c + \frac{c}{\alpha_0^2}\right)\bar{w}^2 + \left(c + \frac{c}{\alpha_0^2}\right)\bar{f}_w^2\bar{w}^2 \\ &\quad + \left(c + \frac{c}{\alpha_0^2}\right)\bar{d}^2 + c\alpha_0^2\bar{\varepsilon}^2 + \frac{c}{\alpha_0^2}\bar{\dot{w}}^2 + 1. \end{aligned} \tag{112}$$

If the following condition is fulfilled

$$\frac{c}{\alpha_0^2}\Delta_\infty^2 + \frac{c}{\alpha_0^2} + c\Delta_\infty^2 < 1 \tag{113}$$

we have

$$m^2 \leq c\|gm\|^2 + \left(c + \frac{c}{\alpha_0^2}\right)\bar{w}^2 + \left(c + \frac{c}{\alpha_0^2}\right)\bar{f}_w^2\bar{w}^2 + \left(c + \frac{c}{\alpha_0^2}\right)\bar{d}^2 + c\alpha_0^2\bar{\varepsilon}^2 + \frac{c}{\alpha_0^2}\bar{\dot{w}}^2 + c. \tag{114}$$

The equation (114) can be rewritten by using the definition of the $\mathcal{L}_{2\delta}$ norm

$$m^2(t) \leq c\int_0^t e^{-\delta(t-\tau)} g^2(\tau) m^2(\tau)\,\mathrm{d}\tau + K, \tag{115}$$

where the definition of K follows directly from (114). By applying Bellman–Gronwall lemma to inequality (115) we get

$$m^2(t) \leq Ke^{-\delta t}e^{c\int_0^t g^2(s)\,\mathrm{d}s} + K\delta\int_0^t e^{-\delta(t-\tau)}e^{c\int_\tau^t g^2(s)\,\mathrm{d}s}\,\mathrm{d}\tau. \tag{116}$$

Because of (110) the following is true

$$c\int_\tau^t g^2(s)\,\mathrm{d}s \leq c_0 + c_1(t-\tau)\Delta_2^2 + c_2(t-\tau)\frac{\bar{d}^2}{m^2} + c_3(t-\tau)\nu_0^2 \tag{117}$$

for $\forall\tau > 0$, $\forall t > \tau$ and some positive constants c_0, c_1, c_2 and c_3. If

$$c_1\Delta_2^2 + c_2\frac{\bar{d}^2}{m^2} + c_3\nu_0^2 \le \delta_0 \qquad (118)$$

then it follows from (116) that $m(t)$ is bounded. The second term becomes arbitrary small as soon as $y_p(t)$ (which is smaller than $m(t)$ by design) reaches some level that depends on the upper bound of the disturbance. That term can be left out and the condition (118) then becomes

$$c_1\Delta_2^2 + c_3\nu_0^2 < \delta_0. \qquad (119)$$

As indicated above, $m(t)$ will be bounded if inequality (119) is satisfied and $m(t)$ is large enough (this is true if $y_p(t)$ is also large enough) so that

$$\frac{\bar{d}^2}{m^2}$$

is negligible. When $m(t)$ falls below the critical value, the system can temporarily become unstable but it stabilizes as soon as (118) is fulfilled again. This phenomenon is the well-known bursting.

Inequality (119) bounds the selection of proper δ_0 in the adaptive law from below. On the other hand δ_0 should not be too large since some transfer functions have to be analytical in the part of the complex plane where

$$\mathrm{Re}[s] \ge -\frac{\delta_0}{2}.$$

The only task that remains unsolved is to show the convergence of the tracking error. Owing to (23) the tracking error equals

$$e = \varepsilon + G_m(p)(\varepsilon n_s^2). \qquad (120)$$

The input to the reference model εn_s^2 can be written as a product of εn_s that belongs to

$$\mathcal{S}\left(\Delta_2^2 + \frac{\bar{d}^2}{m^2} + \nu_0^2\right)$$

and n_s that was shown to be bounded. Therefore it can be concluded:

$$\varepsilon n_s^2 \in \mathcal{S}\left(\Delta_2^2 + \frac{\bar{d}^2}{m^2} + \nu_0^2\right). \qquad (121)$$

If the impulse response of the linear system $H(p)$ belongs to $\mathcal{L}_1$ then $u' \in \mathcal{S}(\mu)$ implies that $y' \in \mathcal{S}(\mu)$ and $y' \in \mathcal{L}_\infty$ for any finite $\mu \ge 0$ where u' and y' are the input and the output of the system $H(p)$, respectively (Ioannou and Sun 1996). In our case the impulse response of the reference model is $b_m e^{-a_m t}$ and therefore it belongs to $\mathcal{L}_1$. Using this fact and (121) it follows:

$$G_m(p)(\varepsilon n_s^2) \in \mathcal{S}\left(\Delta_2^2 + \frac{\bar{d}^2}{m^2} + \nu_0^2\right)$$
$$G_m(p)(\varepsilon n_s^2) \in \mathcal{L}_\infty. \qquad (122)$$

It was shown previously that

$$\varepsilon \in \mathcal{S}\left(\Delta_2^2 + \frac{\bar{d}^2}{m^2} + \nu_0^2\right)$$
$$\varepsilon \in \mathcal{L}_\infty. \qquad (123)$$

By combining (120), (122) and (123) we arrive to the final result

$$e \in \mathcal{S}(\Delta_2^2 + \bar{d}^2 + \nu_0^2)$$
$$e \in \mathcal{L}_\infty, \qquad (124)$$

where it was taken into account that m is bounded.

The proof of (50) follows directly from the proof of Theorem 1 (see equation 27) by noting that

$$\left|\frac{w}{m}\right| < 1 \quad \text{and} \quad \left|\frac{y_p}{m}\right| < 1. \qquad \square$$

References

Anderson, B. D. O., 1985, Adaptive systems, lack of persistency of excitation and bursting phenomena. *Automatica*, **21,** 247–258.

Åström, K. J., and Wittenmark, B., 1995, *Adaptive Control*, 2nd edn (Reading, MA: Addison-Wesley).

Hu, Z., and Lu, X., 1998, Adaptive observer and nonlinear control strategy for chemical reactors via neural networks. *International Journal of Systems Science*, **29,** 915–919.

Ioannou, P. A., and Datta, A., 1991, Robust adaptive control: a unified approach. *Proceedings of the IEEE*, **79**, 1736–1768.

Ioannou, P. A., and Sun, J., 1996, *Robust Adaptive Control* (Englewood Cliffs: Prentice-Hall).

Jagannathan, S., Lewis, F. L., and Pastravanu, O., 1994, Model reference adaptive control of nonlinear dynamical systems using multilayer neural networks. *Proceeding of IEEE International Conference on Neural Networks*, New York, Vol. 7, pp. 4766–4771.

Krstić, M., Kanellakopoulos, I., and Kokotović, P., 1995, *Nonlinear and Adaptive Control Design* (Chichester: Wiley).

Monopoli, R. V., 1974, Model reference adaptive control with an augmented error signal. *IEEE Transactions on Automatic Control*, **AC-19,** 474–484.

Narendra, K. S., and Annaswamy, A. M., 1987, A new adaptive law for robust adaptation without persistent excitation. *IEEE Transactions on Automatic Control*, **AC-32**, 134–145.

Narendra, K. S., Lin, Y. H., and Valavani, L. S., 1980, Stable adaptive controller design, Part II: proof of stability. *IEEE Transactions on Automatic Control*, **AC-25**, 440–448.

Rohrs, C. E., Valavani, L., Athans, M., and Stein, G., 1985, Robustness of continuous-time adaptive control algorithms in the presence of unmodeled dynamics. *IEEE Transactions on Automatic Control*, **AC-30,** 881–889.

Rey, G. J., Johnson, C. R., and Dasgupta, S., 1989, On tuning leakage for performance-robust adaptive control. *IEEE Transactions on Automatic Control*, **34,** 1068–1071.

Rigatos, G. G., Tzafestas, C. S., and Tzafestas, S. G., 2000, Mobile robot motion control in partially unknown environments using a sliding-mode fuzzy-logic controller. *Robotics and Autonomous Systems*, **33,** 1–11.

Rigatos, G. G., Tzafestas, S. G., and Evangelidis, G. J., 2001, Reactive parking control of nonholonomic vehicles via a fuzzy learning automaton. *IEE Proceedings—Control Theory and Applications*, **148,** 169–179.

Spooner, J. T., and Passino, K. M., 1996, Stable adaptive control using fuzzy systems and neural networks. *IEEE Transactions on Fuzzy Systems*, **4,** 339–359.

Škrjanc, I., Kavšek-Biasizzo, K., and Matko, D., 1997, Real-time fuzzy adaptive control. *Engineering Applications of Artificial Intelligence*, **10,** 53–61.

Škrjanc, I., and Matko, D., 1997, Fuzzy adaptive control versus model reference adaptive control of mutable processes. In S. G. Tzafestas (ed.), *Methods and Applications of Intelligent Control* (Dordrecht: Kluwer), pp. 197–216.

Škrjanc, I., and Matko, D., 2000, Predictive functional control based on fuzzy model for heat-exchanger pilot plant. *IEEE Transactions on Fuzzy Systems*, **8**, 705–712.

Takagi, T., and Sugeno, M., 1985, Fuzzy identification of systems and its applications to modelling and control. *IEEE Transactions on Systems, Man, and Cybernetics*, **SMC-15,** 116–132.

Tsakalis, K. S., and Ioannou, P. A., 1987, Adaptive control of linear time-varying plants. *Automatica*, **23,** 459–468.

Tzafestas, S. G., Rigatos, G. G., and Vagelatos, G. A., 2001, Design of fuzzy gain-scheduled robust controllers using Kharitonov's extremal gain margin theorem. *Journal of Intelligent and Fuzzy Systems*, **10,** 39–56.

Vidyasagar, M., 1993, *Nonlinear Systems Analysis*, 2nd edn (Englewood Cliffs: Prentice-Hall).

Wang, L. X., 1993, Stable adaptive fuzzy control of nonlinear systems. *IEEE Transactions on Fuzzy Systems*, **1,** 146–155.